Solving QBF with Counterexample Guided Refinement

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- an extension of SAT with quantifiers
- PSPACE-complete
- formal verification
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- we consider prenex form with maximal blocks of variables

$$Q X_1\ \bar{Q} Y_1\ Q X_2\ \bar{Q} Y_2 \ldots .\ \phi$$

$$\textit{where } Q \in \{\exists, \forall\}$$
$$\bar{\exists} = \forall, \bar{\forall} = \exists$$

A QBF as a Game

- it is useful to think about a QBF as a game between the universal and existential player
- universal player wins when the matrix becomes false
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$$\forall y_1 y_2 \exists x_1 x_2 .\ (y_1 \leftrightarrow x_1) \wedge (y_2 \leftrightarrow x_2)$$

- $\exists$ always wins by playing $x_1 = y_1$, $x_2 = y_2$

Semantics with Winning Move

winning move, base case $QX.\phi$, for ϕ propositional

- for $Q = \exists$, an assignment that makes ϕ true (model of ϕ)
- for $Q = \forall$, an assignment that makes ϕ false

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- an assignment τ s.t. there is no winning move for $\bar{Q}$ for $\Phi[\tau]$

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countermove, for $QX.\ \Phi$, for Φ QBF

- an assignment μ is a countermove to the assignment τ if μ is a winning move for $\bar{Q}$ for $\Phi[\tau]$

Winning Move Semantics

QBF semantics

- $\exists X.\Phi$ is true if and only if there is a winning move for $\exists$
- $\forall X.\Phi$ is false if and only if there is a winning move for $\forall$

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Example

$$\forall y \exists x.\ x \wedge (y \vee \bar{x})$$

- $\{\bar{y}\}$ is a winning move for $\forall$, formula is false
- $\{y\}$ is not a winning move and $\{x\}$ is a countermove

Computing a Winning Move—Base Case

`Solve` ($\exists X.\ \phi$), where ϕ is a propositional
output : a winning move for $\exists$ if there is one; NULL otherwise
return `SAT`(ϕ)

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$\texttt{Solve}$ ($\forall X.\ \phi$), where ϕ is a propositional
output : a winning move for $\forall$ if there is one; NULL otherwise
return $\texttt{SAT}(\neg\phi)$

Naive Algorithm for a Winning Move

```
Function Solve (QX. Φ)
Λ ← {true, false}^X                  // consider all assignments
while true do
    if Λ = ∅ then
        return NULL                  // all assignments used up
    τ ← pick(Λ)                      // pick a candidate solution
    μ ← Solve(Φ[τ])                  // find a countermove
    if μ = NULL then
        return τ                     // winning move
    Λ ← Λ ∖ {τ}                      // remove bad candidate
end
```

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Observation

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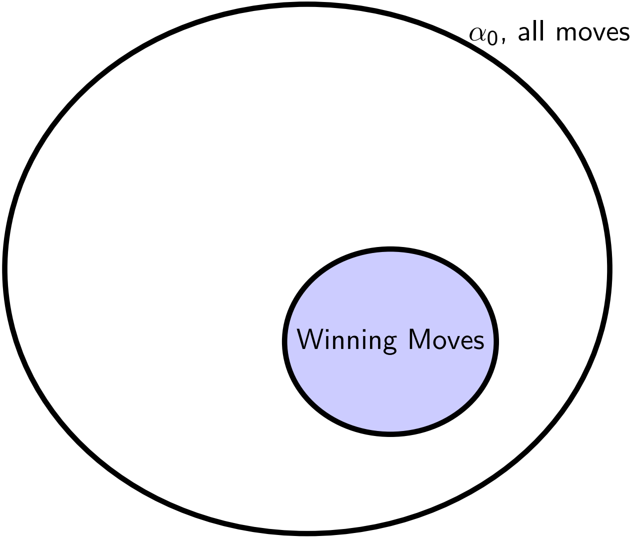
How?

- represent the set of considered candidates as the set of winning moves of a (simpler) QBF (abstraction)
- each time a countermove is found, strengthen the abstraction so that the same countermove cannot be used in the future (refinement)

CounterExample Guided Abstraction Refinement (CEGAR)



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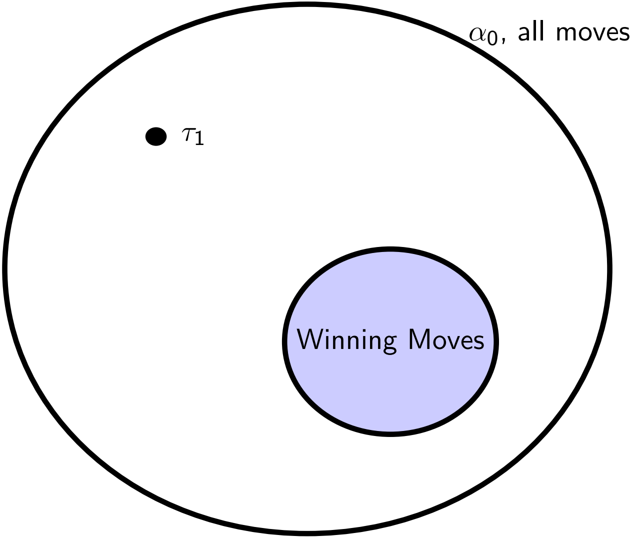

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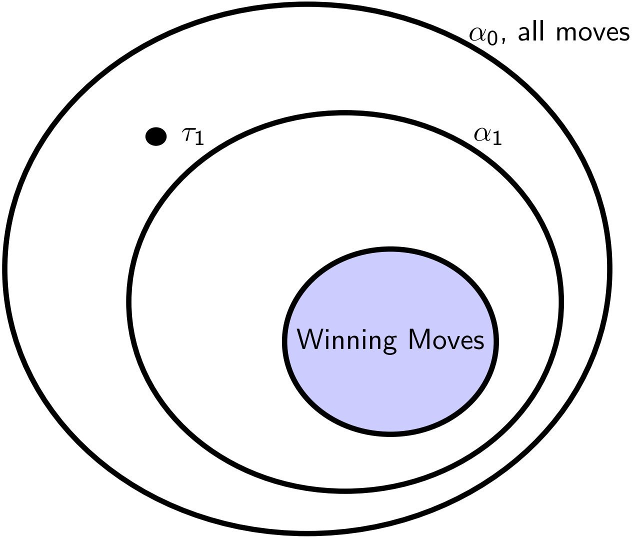

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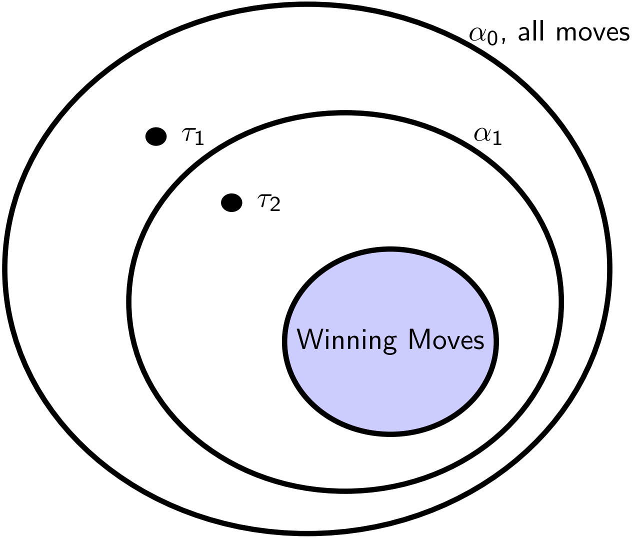

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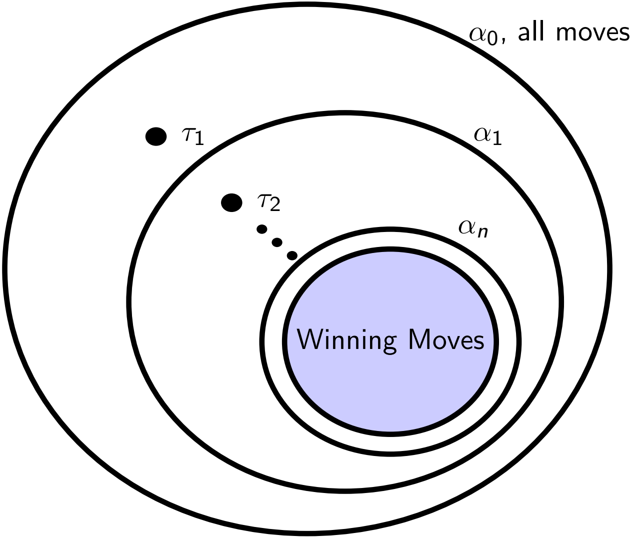

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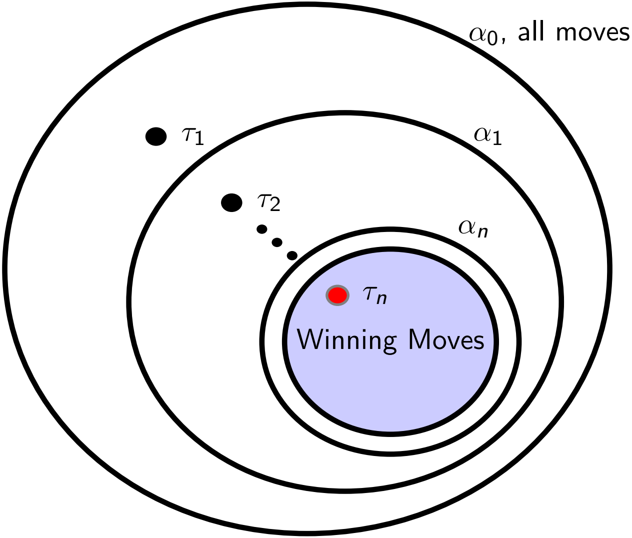

Refinement

for a bad candidate τ

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for a set of countermoves $\omega = \{\mu_1, \ldots, \mu_n\}$

- $\bigwedge_{\mu \in \omega} \Phi[\mu]$, $Q = \exists$
- $\bigvee_{\mu \in \omega} \Phi[\mu]$, $Q = \forall$

Refinement—Example

- $\forall y \exists x.\ x \wedge (y \vee \bar{x})$
- candidate: $\{y\}$, countermove: $\{x\}$
- abstraction: $\forall y.\ y$
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  (with the single winning move $\{\bar{y}_1, \bar{y}_2\}$)

Abstraction-Based Algorithm for a Winning Move

```
Function Solve (QX. Φ)
begin
    if Φ has no quant then
        return (Q = ∃) ? SAT(φ) : SAT(¬φ)
    ω ← ∅
    while true do
        α ← (Q = ∃) ? ⋀_{μ∈ω} Φ[μ] : ⋁_{μ∈ω} Φ[μ]   // abstraction
        τ' ← Solve(Prenex(QX. α))                   // find a candidate
        if τ' = NULL then return NULL               // no winning move
        τ ← {l | l ∈ τ' ∧ var(l) ∈ X}               // filter a move for X
        μ ← Solve(Φ[τ])                             // find a countermove
        if μ = NULL then return τ                   // winning move
        ω ← ω ∪ {μ}                                 // refine
    end
end
```

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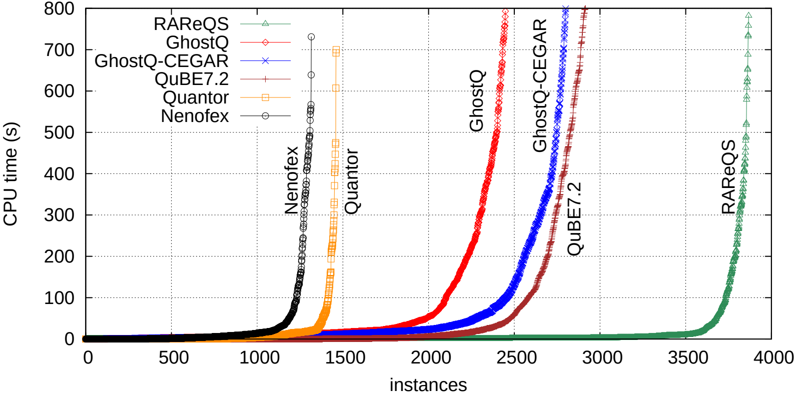
Results for planning and Formal Verification families



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- in the future we plan to further develop the integration between DPLL and CEGAR
- in RAReQS we plan to investigate how to integrate techniques used in other solvers (e.g. dependency detection)

Questions?

Results

Total instances solved (out of 4669):

RAReQS	GhostQ	GhostQ-C	Qube	Quantor	Nenofex
3868	2449	2801	2916	1462	1317