# On Deciding MUS Membership with QBF

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# CNF and Unsatisfiability

$$\{\, x \vee y,\ \ \neg y,\quad \neg x,\quad z \,\}$$

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## MUS-Membership

IN: a clause $\omega$ and a CNF $\phi$

Q: Is there an MUS $\psi \subseteq \phi$ such that $\omega \in \psi$?

# Motivation

## Restoring Consistency

Removing a clause that is not part of any MUS, will certainly not restore consistency.

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## Product Configuration

When configuring a product, some sets of its features result in an inconsistent configuration. Clearly, it is useful for the user(s) to know if a feature is relevant for the inconsistency.

# How Hard Is It?

$$
\begin{array}{lll}
\{\ x_1, & & x_1 \rightarrow z, \\
\ \ x_2, & & x_2 \rightarrow z, \\
\ \ y_1, & & y_1 \rightarrow \neg z, \\
\ \ y_2, & & y_2 \rightarrow \neg z, \\
\ \ \omega & \} &
\end{array}
$$

# How Hard Is It?

$$\{\ \boxed{x_1,} \qquad \boxed{x_1 \to z,}$$

$$x_2, \qquad x_2 \to z,$$

$$\boxed{y_1,} \qquad \boxed{y_1 \to \neg z,}$$

$$y_2, \qquad y_2 \to \neg z,$$

$$\omega \quad \}$$

# How Hard Is It?

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$$\omega \qquad \}$$

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MUS-MEMBERSHIP is $\Sigma_2^P$-complete [Kul07]

# Approaches to the Problem

$$\begin{tikzcd}
\mathrm{QBF}_{3,\exists},\ O(n) & \boxed{\text{MUS-membership}} \arrow[l] \arrow[r] \arrow[d] & \mathrm{QBF}_{2,\exists},\ O(n^2) \\
& \text{MSS-membership} \arrow[dl, bend left] \arrow[dr, bend right] & \\
\begin{array}{c}\mathrm{QBF}_{2,\exists},\ O(n) \\ \text{[JMS11]}\end{array} & & \begin{array}{c}\text{Circ-Infer},\ O(n) \\ \text{[JGMS10]}\end{array}
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## Relaxation

$$\phi^* = \{c \vee r_c \mid c \in \phi\}$$

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## Relaxing Clauses Example

- $\phi = \{x \vee y, \neg x, \neg y\}$
- $\phi^* = \{r_1 \vee x \vee y, r_2 \vee \neg x, r_3 \vee \neg y\}$

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$$\begin{array}{l|l} r_1 = 0 & r_1 \vee \ x \vee y \\ r_2 = 0 & r_2 \vee \ \neg x \\ r_3 = 1 & r_3 \vee \neg y \end{array}$$

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## Subset

$R = \{r_1, \dots, r_n\},\ R' = \{r'_1, \dots, r'_n\}$

$$R < R' \equiv \bigwedge_{r_i \in R} r_i \Rightarrow r'_i \wedge \bigvee_{r_i \in R} \neg r_i \wedge r'_i$$

# Naïve Approaches

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**exists** $\psi \subseteq \phi$ s.t. $\omega \in \psi$ **and** $\psi$ is unsatisfiable **and** **forall** $\psi' \subsetneq \psi$ is satisfiable

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$\exists R.\ \ \neg r_\omega \wedge (\forall X.\neg\phi^*(R,X)) \wedge (\forall R'.(R < R') \Rightarrow \exists X'.\phi^*(R',X'))$

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## 2-level quantification, $O(n^2)$

$\exists R.\ \neg r_\omega \wedge (\forall X. \neg \phi^*(R, X)) \wedge$
$\bigwedge_{r_{\omega_i} \in R} (\neg r_{\omega_i} \Rightarrow \exists X^{\omega_i}.\phi^*[r_{\omega_i}/1](R, X^{\omega_i}))$

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## 2-level quantification, $O(n^2)$, prefix form

$$\exists R X^{\omega_1} \ldots \exists X^{\omega_n} \forall X.\ \ \neg r_\omega \wedge \neg \phi^*(R, X) \wedge \bigwedge_{r_{\omega_i} \in R} (\neg r_{\omega_i} \Rightarrow \phi^*[r_{\omega_i}/1](R, X^{\omega_i}))$$

# Approaches to the Problem

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# From MUS-MEMBERSHIP to MSS-MEMBERSHIP

## MSS

A set of clauses $\psi \subseteq \phi$ is a Maximally Satisfiable Subset (MSS) iff $\psi$ is satisfiable and any set $\psi' \subseteq \phi$ such that $\psi \subsetneq \psi'$ is unsatisfiable.

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## MSS-MEMBERSHIP

IN: A CNF formula $\phi$ and a clause $\omega \in \phi$.

Q: Is there an MSS $\psi$ of $\phi$ such that $\omega \notin \psi$?

### MUS-Membership $\leftrightarrow$ MSS-membership

A clause $\omega$ belongs to some MUS iff there is some MSS that does not contain $\omega$.

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### MSS-MEMBERSHIP to QBF

$\exists R \exists X \forall R' \forall X'.\ r_\omega \wedge \phi^*(R, X) \wedge (R' < R \Rightarrow \neg\phi^*(R', X'))$

# Minimal Models

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$$\begin{tikzcd}
 & {[1,1,1]} & \\
{[1,1,0]} \arrow[ur] & {[1,0,1]} \arrow[u] & {[0,1,1]} \arrow[ul] \\
{[1,0,0]} \arrow[u] \arrow[ur] & {[0,1,0]} \arrow[ul] \arrow[ur] & {[0,0,1]} \arrow[ul] \arrow[u] \\
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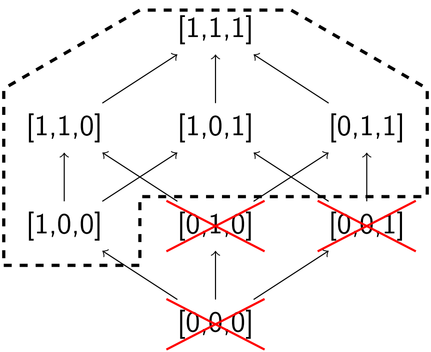

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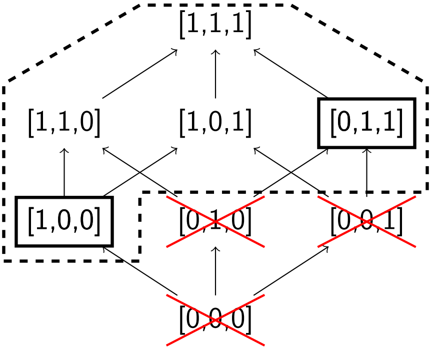

# Entailment in Circumscription

**CircInfer**

IN: $\tau$ and $\psi$ be propositional formulas

Q: Does $\psi$ hold in all minimal models of $\tau$.

$$\tau \models_{\min} \psi$$

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## CIRCINFER, complexity

- Deciding $\tau \models_{\min} \psi$ is in $\Pi_2^P$-complete [EG93]

# MSSes and Minimal Models

$$\phi = \{x, \neg x, z\} \qquad \begin{array}{ccr} r_1 & \ldots & x \\ r_2 & \ldots & \neg x \\ r_3 & \ldots & z \end{array}$$

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 & \{\}, [1,1,1] & \\
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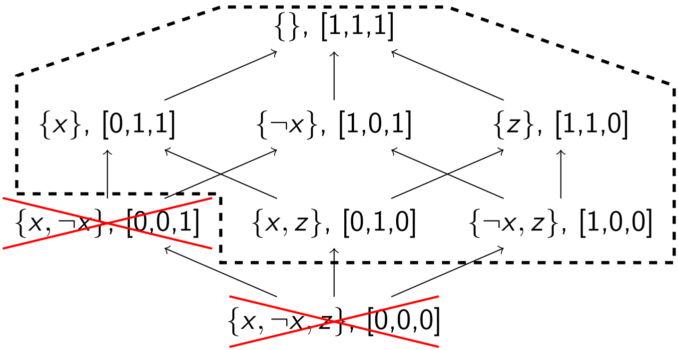

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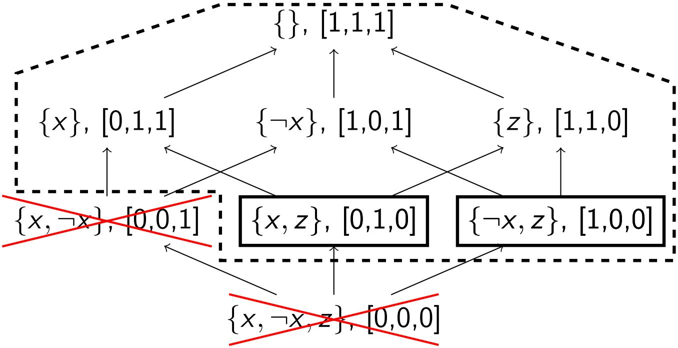

# From MSS-Membership to CircInfer

**MSSes $\leftrightarrow$ Min. Models**

MSSes correspond to $R$-minimal models of $\phi^*(R, X)$.

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### MSSes $\leftrightarrow$ Min. Models

MSSes correspond to $R$-minimal models of $\phi^*(R, X)$.

### MUS-Membership $\leftrightarrow$ MSS-Membership $\leftrightarrow$ CircInfer

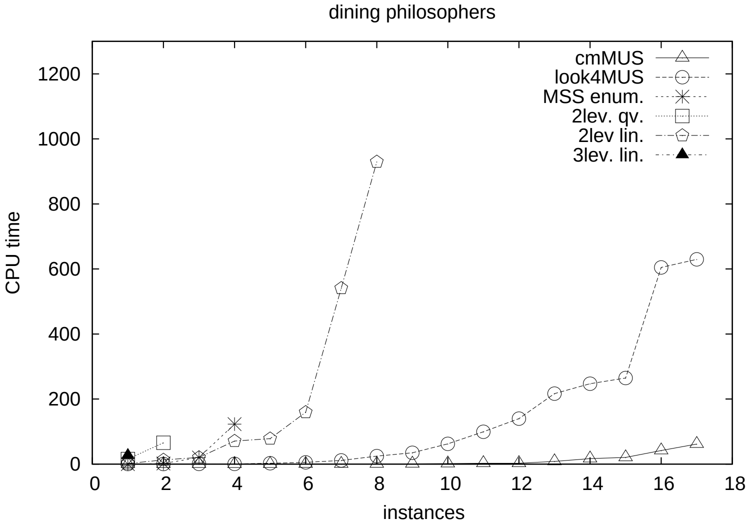
A clause $\omega$ belongs to some MUS of $\phi$ iff there exists a $R$-minimal model $M$ of $\phi^*$ such that $M \models r_\omega$, equivalently:

$$\phi^* \not\models_R^{circ} \neg r_\omega$$

| | `cmMUS` | `look4MUS` | MSS enum. | 2lev. lin. |
|---|---|---|---|---|
| Nemesis (223) | **223** | **223** | 31 | 29 |
| DC (84) | 46 | 13 | **49** | 36 |
| dining phil. (22) | **17** | **17** | 4 | 8 |
| dimacs (87) | **87** | 82 | 51 | 51 |
| ezfact (41) | **20** | 11 | 11 | 10 |
| total (457) | **393** | 346 | 146 | 134 |

| | 2lev. qv. | 3lev. lin. (`QuBE`) | 3lev. lin. (`sSolve`) |
|---|---|---|---|
| Nemesis (223) | 9 | 13 | 0 |
| DC (84) | 0 | 4 | 0 |
| dining phil. (22) | 2 | 1 | 0 |
| dimacs (87) | 18 | 25 | 4 |
| ezfact (41) | 0 | 0 | 0 |
| total (457) | 29 | 43 | 4 |

# Results



dining philosophers
CPU time
instances
cmMUS
look4MUS
MSS enum.
2lev. qv.
2lev lin.
3lev. lin.
0
200
400
600
800
1000
1200
0
2
4
6
8
10
12
14
16
18

# Summary

$$
\begin{tikzcd}
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Thomas Eiter and Georg Gottlob.
Propositional circumscription and extended closed-world reasoning are $\Pi_2^P$-complete.
*Theor. Comput. Sci.*, 114(2):231–245, 1993.

Mikoláš Janota, Radu Grigore, and Joao Marques-Silva.
Counterexample guided abstraction refinement algorithm for propositional circumscription.
In *JELIA '10*, 2010.

Mikoláš Janota and Joao Marques-Silva.
Abstraction-based algorithm for 2QBF.
In *SAT*, 2011.

Oliver Kullmann.
Constraint satisfaction problems in clausal form: Autarkies and minimal unsatisfiability.
*ECCC*, 14(055), 2007.