Modern SAT Solving

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CP 2007
Motivation - Why SAT?

- Boolean Satisfiability (SAT) has seen significant improvements in recent years
  - Ok, but SAT is simply a subset of CP...
    - This does not make SAT a simple issue!
  - So, can we learn anything from there?
    - Much more than you can imagine!
Motivation - Some lessons from SAT I

- **Time is everything**
  - Good ideas are not enough, you have to be fast!
  - One thing is the algorithm, another thing is the implementation
  - Make your source code available
    - Otherwise people will have to wait for years before realising what you have done
Motivation - Some lessons from SAT II

• Competitions are essential
  – To check the state-of-the-art
  – To keep the community alive
  – To get students involved
Motivation - Some lessons from SAT III

- There is no perfect solver!
  - Do not expect your solver to beat all the other solvers on all problem instances
- What makes a good solver?
  - Correctness and robustness for sure...
  - Being most often the best for its category: industrial, handmade or random
  - Being able to solve instances from different problems
• Get all the info from the SAT competition web page
  – Organizers, judges, benchmarks, executables, source code
  – Winners
    ▶ Industrial, handmade and random benchmarks
    ▶ Sat+Unsat, Sat, Unsat categories
    ▶ Gold, Silver, Bronze medals

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<thead>
<tr>
<th>Organizing committee</th>
<th>Daniel Le Berre, Olivier Roussel and Laurent Simon</th>
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<tr>
<td>Judges</td>
<td>Ewald Speckenmeyer, Geoff Sutcliffe and Linhao Zhang</td>
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<td>Benchmarks</td>
<td>random (tar.gz 4 MB), crafted (tar, gzip compressed files inside 175 MB), industrial (tar, gzip compressed files inside, 356 MB) + velov's VLIW-SAT 4.0 and VLIW-UNSAT 2.0 + IBM benchmarks</td>
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SAT 2007 competition

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<tr>
<th>Systems</th>
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All winners precompiled for Linux (tgz, 25/10 MB). Source code (competition division only, tgz, updated 11/7/07-08/08).
Outline

What is Boolean Satisfiability?

Applications

Modeling

Algorithms
  Fundamentals
  Local Search
  The DPLL Algorithm
  Conflict-Driven Clause Learning (CDCL)

Extensions
Outline
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Extensions
Boolean Formulas

- Boolean formula $\varphi$ is defined over a set of propositional variables $x_1, \ldots, x_n$, using the standard propositional connectives $\neg, \land, \lor, \rightarrow, \leftrightarrow$, and parenthesis
  - The domain of propositional variables is $\{0, 1\}$
  - Example: $\varphi(x_1, \ldots, x_3) = ((\neg x_1 \land x_2) \lor x_3) \land (\neg x_2 \lor x_3)$

- A formula $\varphi$ in conjunctive normal form (CNF) is a conjunction of disjunctions (clauses) of literals, where a literal is a variable or its complement
  - Example: $\varphi(x_1, \ldots, x_3) = (\neg x_1 \lor x_3) \land (x_2 \lor x_3) \land (\neg x_2 \lor x_3)$

- Can encode any Boolean formula into CNF (more later)
Boolean Satisfiability (SAT)

- The Boolean satisfiability (SAT) problem:
  - Find an assignment to the variables $x_1, \ldots, x_n$ such that $\varphi(x_1, \ldots, x_n) = 1$, or prove that no such assignment exists.

- SAT is an **NP-complete** decision problem [Cook’71]
  - SAT was the first problem to be shown NP-complete.
  - There are **no** known polynomial time algorithms for SAT.
  - 36-year old conjecture: Any algorithm that solves SAT is exponential in the number of variables, in the worst-case.
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Applications of SAT I

- **Formal methods:**
  - Hardware model checking; Software model checking; Termination analysis of term-rewrite systems; Test pattern generation (testing of software & hardware); etc.

- **Artificial intelligence:**
  - Planning; Knowledge representation; Games (n-queens, sudoku, social golpher’s, etc.)

- **Bioinformatics:**
  - Haplotype inference; Pedigree checking; Comparative genomics; etc.

- **Design automation:**
  - Equivalence checking; Delay computation; Fault diagnosis; Noise analysis; etc.

- **Security:**
  - Cryptanalysis; Inversion attacks on hash functions; etc.
Applications of SAT II

- Computationally hard problems:
  - Graph coloring; Traveling salesperson; etc.

- Mathematical problems:
  - van der Waerden numbers; etc.

- Core engine for other solvers: 0-1 ILP; QBF; #SAT; SMT; ...

- Integrated into theorem provers: HOL; Isabelle; ...
Example: Graph Coloring I

- Decide whether one can assign one of $K$ colors to each of the vertices of graph $G = (V, E)$ such that adjacent vertices are assigned different colors.

Valid coloring

Invalid coloring
Example: Graph Coloring II

- Given $N = |V|$ vertices and $K$ colors, create $N \times K$ variables: $x_{ij} = 1$ iff vertex $i$ is assigned color $j$; 0 otherwise.

- For each edge $(u, v)$, require different assigned colors to $u$ and $v$:

$$1 \leq j \leq K, \quad (\neg x_{uj} \lor \neg x_{vj})$$

- Each vertex is assigned exactly one color:

$$1 \leq i \leq N, \quad \sum_{j=1}^{K} x_{ij} = 1$$
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Extensions
Representing AtLeast, AtMost and Equal Constraints

- How to represent in CNF the constraint $\sum_{j=1}^{N} x_j \geq 1$?
  - Standard solution: $(x_1 \lor \ldots \lor x_N)$

- How to represent in CNF the constraint $\sum_{j=1}^{N} x_{ij} \leq 1$?
  - Naive solution: $\forall_{j_1=1..N} \forall_{j_2=j_1+1..N} (\neg x_{ij_1} \lor \neg x_{ij_2})$
    - Number of clauses grows quadratically with $N$
  - More compact (i.e. linear) solutions possible
    - At the cost of using additional variables

- How to represent in CNF the constraint $\sum_{j=1}^{N} x_{ij} = 1$?
  - Standard solution: one AtMost 1 and one AtLeast 1 constraints
Representing Boolean Circuits / Formulas I

- Satisfiability problems can be defined on Boolean circuits/formulas
- Can represent circuits/formulas as CNF formulas [Tseitin’68]
  - For each (simple) gate, CNF formula encodes the consistent assignments to the gate’s inputs and output
    - Given $z = \text{OP}(x, y)$, represent in CNF $z \leftrightarrow \text{OP}(x, y)$
  - CNF formula for the circuit is the conjunction of CNF formula for each gate

\[
\varphi_c = (a \lor c) \land (b \lor c) \land (\neg a \lor \neg b \lor \neg c)
\]

\[
\varphi_t = (\neg r \lor t) \land (\neg s \lor t) \land (r \lor s \lor \neg t)
\]
\[ \varphi_c = (a \lor c) \land (b \lor c) \land (\neg a \lor \neg b \lor \neg c) \]
Representing Boolean Circuits / Formulas III

- CNF formula for the circuit is the conjunction of the CNF formula for each gate
  - Can specify objectives with additional clauses

\[ \varphi = (a \lor x) \land (b \lor x) \land (\neg a \lor \neg b \lor \neg x) \land (x \lor \neg y) \land (c \lor \neg y) \land (\neg x \lor \neg c \lor y) \land (\neg y \lor z) \land (\neg d \lor z) \land (y \lor d \lor \neg z) \land (z) \]

- Note: \( z = d \lor (c \land (\neg (a \land b))) \)
  - No distinction between Boolean circuits and formulas
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Extensions
Algorithms for SAT

- Incomplete algorithms (i.e. can only prove (un)satisfiability):
  - Local search / hill-climbing
  - Genetic algorithms
  - Simulated annealing
  - ...

- Complete algorithms (i.e. can prove both satisfiability and unsatisfiability):
  - Proof system(s)
    - Natural deduction
    - Resolution
    - Stalmarck’s method
    - Recursive learning
    - ...
  - Binary Decision Diagrams (BDDs)
  - Backtrack search / DPLL
    - Conflict-Driven Clause Learning (CDCL)
  - ...
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Extensions
Definitions

- Propositional variables can be assigned value 0 or 1
  - In some contexts variables may be unassigned

- A clause is satisfied if at least one of its literals is assigned value 1
  \[(x_1 \lor \neg x_2 \lor \neg x_3)\]

- A clause is unsatisfied if all of its literals are assigned value 0
  \[(x_1 \lor \neg x_2 \lor \neg x_3)\]

- A clause is unit if it contains one single unassigned literal and all other literals are assigned value 0
  \[(x_1 \lor \neg x_2 \lor \neg x_3)\]

- A formula is satisfied if all of its clauses are satisfied
- A formula is unsatisfied if at least one of its clauses is unsatisfied
Pure Literals

- A literal is **pure** if only occurs as a positive literal or as a negative literal in a CNF formula
  - Example:
    \[ \varphi = (\neg x_1 \lor x_2) \land (x_3 \lor \neg x_2) \land (x_4 \lor \neg x_5) \land (x_5 \lor \neg x_4) \]
  - \( x_1 \) and \( x_3 \) and pure literals

- **Pure literal rule:**
  Clauses containing pure literals can be removed from the formula (i.e. just assign pure literals to the values that satisfy the clauses)
  - For the example above, the resulting formula becomes:
    \[ \varphi = (x_4 \lor \neg x_5) \land (x_5 \lor \neg x_4) \]

- A reference technique until the mid 90s; nowadays seldom used
Unit Propagation

• **Unit clause rule:**
  Given a unit clause, its only unassigned literal must be assigned value 1 for the clause to be satisfied
  
  - Example: for unit clause \((x_1 \lor \neg x_2 \lor \neg x_3)\), \(x_3\) must be assigned value 0

• **Unit propagation**
  Iterated application of the unit clause rule

\[
(x_1 \lor \neg x_2 \lor \neg x_3) \land (\neg x_1 \lor \neg x_3 \lor x_4) \land (\neg x_1 \lor \neg x_2 \lor x_4)
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- **Unit propagation**
  Iterated application of the unit clause rule:
  \[
  \left((x_1 \lor \neg x_2 \lor \neg x_3) \land (\neg x_1 \lor \neg x_3 \lor x_4) \land (\neg x_1 \lor \neg x_2 \lor x_4)\right) \\
  \left((x_1 \lor \neg x_2 \lor \neg x_3) \land (\neg x_1 \lor \neg x_3 \lor x_4) \land (\neg x_1 \lor \neg x_2 \lor \neg x_4)\right)
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  \]

  \[
  (x_1 \lor \neg x_2 \lor \neg x_3) \land (\neg x_1 \lor \neg x_3 \lor x_4) \land (\neg x_1 \lor \neg x_2 \lor \neg x_4)
  \]

- **Unit propagation** can satisfy clauses but can also unsatisfy clauses. Unsatisfied clauses create conflicts.
Resolution

- Resolution rule:
  - If a formula $\varphi$ contains clauses $(x \lor \alpha)$ and $(\neg x \lor \beta)$, then one can infer $(\alpha \lor \beta)$

$$((x \lor \neg \alpha) \land (\neg x \lor \beta)) \vdash (\alpha \lor \beta)$$

- Resolution forms the basis of a complete algorithm for SAT
  - Iteratively apply the following steps: [Davis&Putnam’60]
    - Select variable $x$
    - Apply resolution rule between every pair of clauses of the form $(x \lor \alpha)$ and $(\neg x \lor \beta)$
    - Remove all clauses containing either $x$ or $\neg x$
    - Apply the pure literal rule and unit propagation
  - Terminate when either the empty clause or the empty formula is derived
Resolution – An Example

\((x_1 \lor \neg x_2 \lor \neg x_3) \land (\neg x_1 \lor \neg x_2 \lor \neg x_3) \land (x_2 \lor x_3) \land (x_3 \lor x_4) \land (x_3 \lor \neg x_4) \vdash\)
Resolution – An Example

\[(x_1 \lor \neg x_2 \lor \neg x_3) \land (\neg x_1 \lor \neg x_2 \lor \neg x_3) \land (x_2 \lor x_3) \land (x_3 \lor x_4) \land (x_3 \lor \neg x_4) \vdash \]

\[(\neg x_2 \lor \neg x_3) \land (x_2 \lor x_3) \land (x_3 \lor x_4) \land (x_3 \lor \neg x_4) \vdash \]
Resolution – An Example

\[(x_1 \lor \neg x_2 \lor \neg x_3) \land (\neg x_1 \lor \neg x_2 \lor \neg x_3) \land (x_2 \lor x_3) \land (x_3 \lor x_4) \land (x_3 \lor \neg x_4) \lor\]

\[(\neg x_2 \lor \neg x_3) \land (x_2 \lor x_3) \land (x_3 \lor x_4) \land (x_3 \lor \neg x_4) \lor\]

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\[(x_3)\]

- Formula is SAT
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Local Search

The DPLL Algorithm

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Extensions
Organization of Local Search

- Local search is incomplete; *usually* it *cannot* prove unsatisfiability
  - Very effective in specific contexts

- Example:

\[(x_1 \lor \neg x_2 \lor \neg x_3) \land (\neg x_1 \lor \neg x_3 \lor x_4) \land (\neg x_1 \lor \neg x_2 \lor x_4)\]
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- Start with (possibly random) assignment:
  \(x_4 = 0, x_1 = x_2 = x_3 = 1\)

- And repeat a number of times:
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  - If not all clauses satisfied, flip variable (e.g. \(x_4\))
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- And repeat a number of times:
  - If not all clauses satisfied, flip variable (e.g. \(x_4\))
  - Done if all clauses satisfied
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- Example:

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- Start with (possibly random) assignment:
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- And repeat a number of times:
  - If not all clauses satisfied, flip variable (e.g. \(x_4\))
  - Done if all clauses satisfied

- Repeat (random) selection of assignment a number of times
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Historical Perspective I

In 1960, M. Davis and H. Putnam proposed the DP algorithm:
- Resolution used to eliminate 1 variable at each step
- Applied the pure literal rule and unit propagation

Original algorithm was inefficient
Historical Perspective I

• In 1960, M. Davis and H. Putnam proposed the DP algorithm:
  – Resolution used to eliminate 1 variable at each step
  – Applied the pure literal rule and unit propagation

• Original algorithm was inefficient
In 1962, M. Davis, G. Logemann and D. Loveland proposed an alternative algorithm:
- Instead of eliminating variables, the algorithm would split on a given variable at each step
- Also applied the pure literal rule and unit propagation

The 1962 algorithm is actually an implementation of backtrack search

Over the years the 1962 algorithm became known as the DPLL (sometimes DLL) algorithm
Basic Algorithm for SAT – DPLL

- Standard backtrack search
- At each step:
  - [DECIDE] Select decision assignment
  - [DEDUCE] Apply unit propagation and (optionally) the pure literal rule
  - [DIAGNOSIS] If conflict identified, then backtrack
    - If cannot backtrack further, return **UNSAT**
    - Otherwise, proceed with unit propagation
  - If formula satisfied, return **SAT**
  - Otherwise, proceed with another decision
An Example of DPLL

\[ \varphi = (a \lor \neg b \lor d) \land (a \lor \neg b \lor e) \land \\
(\neg b \lor \neg d \lor \neg e) \land \\
(a \lor b \lor c \lor d) \land (a \lor b \lor c \lor \neg d) \land \\
(a \lor b \lor \neg c \lor e) \land (a \lor b \lor \neg c \lor \neg e) \]
An Example of DPLL

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(\neg b \lor \neg d \lor \neg e) \land \\
(a \lor b \lor c \lor d) \land (a \lor b \lor c \lor \neg d) \land \\
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conflict
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Outline

What is Boolean Satisfiability?

Applications

Modeling

Algorithms
   Fundamentals
   Local Search
   The DPLL Algorithm
   Conflict-Driven Clause Learning (CDCL)

Extensions
CDCL SAT Solvers

- Introduced in the 90's
  [Marques-Silva&Sakallah’96][Bayardo&Schrag’97]
- Inspired on DPLL
  - Must be able to prove both satisfiability and unsatisfiability
- New clauses are learnt from conflicts
- Structure of conflicts exploited (UIPs)
- Backtracking can be non-chronological
- Efficient data structures [Moskewicz&al’01]
  - Compact and reduced maintenance overhead
- Backtrack search is periodically restarted [Gomes&al’98]

Can solve instances with hundreds of thousand variables and tens of million clauses
CDCL SAT Solvers

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- Can solve instances with hundreds of thousand variables and tens of million clauses
Clause Learning

- During backtrack search, for each conflict learn new clause, which explains and prevents repetition of the same conflict.

\[ \varphi = (a \lor b) \land (\neg b \lor c \lor d) \land (\neg b \lor e) \land (\neg d \lor \neg e \lor f) \ldots \]
Clause Learning

- During backtrack search, for each conflict learn new clause, which explains and prevents repetition of the same conflict

\[
\varphi = (a \lor b) \land (\neg b \lor c \lor d) \land (\neg b \lor e) \land (\neg d \lor \neg e \lor f) \ldots
\]

- Assume decisions \(c = 0\) and \(f = 0\)
Clause Learning

- During backtrack search, for each conflict learn a new clause, which explains and prevents repetition of the same conflict

\[ \varphi = (a \lor b) \land (\lnot b \lor c \lor d) \land (\lnot b \lor e) \land (\lnot d \lor \lnot e \lor f) \ldots \]

- Assume decisions $c = 0$ and $f = 0$
- Assign $a = 0$ and imply assignments
Clause Learning

- During backtrack search, for each conflict learn new clause, which explains and prevents repetition of the same conflict

\[ \varphi = (a \lor b) \land (\neg b \lor c \lor d) \land (\neg b \lor e) \land (\neg d \lor \neg e \lor f) \ldots \]

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- A conflict is reached: $(\neg d \lor \neg e \lor f)$ is unsatisfied
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- Assume decisions \( c = 0 \) and \( f = 0 \)
- Assign \( a = 0 \) and imply assignments
- A conflict is reached: \( (\neg d \lor \neg e \lor f) \) is unsatisfied
- \( (a = 0) \land (c = 0) \land (f = 0) \Rightarrow (\varphi = 0) \)
Clause Learning

- During backtrack search, for each conflict learn new clause, which explains and prevents repetition of the same conflict

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- \( (a = 0) \land (c = 0) \land (f = 0) \Rightarrow (\varphi = 0) \)
- \( (\varphi = 1) \Rightarrow (a = 1) \lor (c = 1) \lor (f = 1) \)
Clause Learning

- During backtrack search, for each conflict learn new clause, which explains and prevents repetition of the same conflict

\[ \varphi = \left( a \lor b \right) \land \left( \neg b \lor c \lor d \right) \land \left( \neg b \lor e \right) \land \left( \neg d \lor \neg e \lor f \right) \ldots \]

- Assume decisions \( c = 0 \) and \( f = 0 \)
- Assign \( a = 0 \) and imply assignments
- A conflict is reached: \( \left( \neg d \lor \neg e \lor f \right) \) is unsatisfied
- \( (a = 0) \land (c = 0) \land (f = 0) \Rightarrow (\varphi = 0) \)
- \( (\varphi = 1) \Rightarrow (a = 1) \lor (c = 1) \lor (f = 1) \)
- Learn new clause \( (a \lor c \lor f) \)
Non-Chronological Backtracking

- During backtrack search, for each conflict backtrack to one of the causes of the conflict

\[
\varphi = (a \lor b) \land (\neg b \lor c \lor d) \land (\neg b \lor e) \land (\neg d \lor \neg e \lor f) \land (a \lor c \lor f) \land (\neg a \lor g) \land (\neg g \lor b) \land (\neg h \lor j) \land (\neg i \lor k)
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- Assume decisions \( c = 0, f = 0, h = 0 \) and \( i = 0 \)
Non-Chronological Backtracking

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- Assume decisions \( c = 0, \ f = 0, \ h = 0 \) and \( i = 0 \)
- Assignment \( a = 0 \) caused conflict \( \Rightarrow \) learnt clause \( (a \lor c \lor f) \) implies \( a = 1 \)
Non-Chronological Backtracking

- During backtrack search, for each conflict backtrack to one of the causes of the conflict

\[ \varphi = (a \lor b) \land (\neg b \lor c \lor d) \land (\neg b \lor e) \land (\neg d \lor \neg e \lor f) \land (a \lor c \lor f) \land (\neg a \lor g) \land (\neg g \lor b) \land (\neg h \lor j) \land (\neg i \lor k) \]

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- Assume decisions \( c = 0, f = 0, h = 0 \) and \( i = 0 \)
- Assignment \( a = 0 \) caused conflict \( \Rightarrow \) learnt clause \((a \lor c \lor f)\) implies \( a = 1 \)
- A conflict is again reached: \((\neg d \lor \neg e \lor f)\) is unsatisfied
Non-Chronological Backtracking

- During backtrack search, for each conflict backtrack to one of the causes of the conflict

\[
\varphi = (a \lor b) \land (\neg b \lor c \lor d) \land (\neg b \lor e) \land (\neg d \lor \neg e \lor f) \land (a \lor c \lor f) \land (\neg a \lor g) \land (\neg g \lor b) \land (\neg h \lor j) \land (\neg i \lor k)
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- Assume decisions \( c = 0 \), \( f = 0 \), \( h = 0 \) and \( i = 0 \)
- Assignment \( a = 0 \) caused conflict \( \Rightarrow \) learnt clause \( (a \lor c \lor f) \) implies \( a = 1 \)
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- \( (c = 0) \land (f = 0) \Rightarrow (\varphi = 0) \)
Non-Chronological Backtracking

- During backtrack search, for each conflict backtrack to one of the causes of the conflict

\[ \varphi = (a \lor b) \land (\neg b \lor c \lor d) \land (\neg b \lor e) \land (\neg d \lor \neg e \lor f) \land (a \lor c \lor f) \land (\neg a \lor g) \land (\neg g \lor b) \land (\neg h \lor j) \land (\neg i \lor k) \]

- Assume decisions \( c = 0, f = 0, h = 0 \) and \( i = 0 \)
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- Assignment \( a = 0 \) caused conflict \( \Rightarrow \) learnt clause \((a \lor c \lor f)\) implies \( a = 1 \)
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- \((\varphi = 1) \Rightarrow (c = 1) \lor (f = 1)\)
- Learn new clause \((c \lor f)\)
Non-Chronological Backtracking

\[ (a \lor c \lor f) \]  \[ (c \lor f) \]
Non-Chronological Backtracking

- Learnt clause: \((c \lor f)\)
- Need to backtrack, given new clause
- Backtrack to most recent decision: \(f = 0\)

Clause learning and non-chronological backtracking are hallmarks of modern SAT solvers.
Most Recent Backtracking Scheme

\[(a \lor c \lor f)\]
Most Recent Backtracking Scheme

\[(a \lor c \lor f)\]
Most Recent Backtracking Scheme

- Learnt clause: $(a \lor c \lor f)$
- No need to assign $a = 1$ - backtrack to most recent decision: $f = 0$
- Search algorithm is no longer a traditional backtracking scheme
Unique Implication Points (UIPs)

- Exploit **structure** from the implication graph
  - To have a more aggressive backtracking policy
- Identify **additional clauses** to be learnt [Marques-Silva&Sakallah’96]
  - Create clauses \( (a \lor c \lor f) \) and \( (\neg i \lor f) \)
  - Imply not only \( a = 1 \) but also \( i = 0 \)
- 1st UIP scheme is the most efficient [Zhang&al’01]
  - Create only one clause \( (\neg i \lor f) \)
  - Avoid creating similar clauses involving the same literals
Clause deletion policies

- Keep only the **small clauses** [Marques-Silva&Sakallah’96]
  - For each conflict record one clause
  - Keep clauses of size \( \leq K \)
  - Large clauses get deleted when become unresolved
- Keep only the **relevant clauses** [Bayardo&Schrag’97]
  - Delete unresolved clauses with \( \leq M \) free literals
- Keep only the clauses **that are used** [Goldberg&Novikov’02]
  - Keep track of clauses activity
Data Structures

- **Key point:** only unit and unsatisfied clauses *must* be detected during search
  - Formula is **unsatisfied** when at least one clause is unsatisfied
  - Formula is **satisfied** when all the variables are assigned and there are no unsatisfied clauses

- **In practice:** unit and unsatisfied clauses may be identified using only two references

- **Standard data structures** *(adjacency lists):*
  - Each variable $x$ keeps a reference to all clauses containing a literal in $x$

- **Lazy data structures** *(watched literals):*
  - For each clause, only two variables keep a reference to the clause, i.e. only 2 literals are *watched*
Each variable $x$ keeps a reference to all clauses containing a literal in $x$
- If variable $x$ is assigned, then all clauses containing a literal in $x$ are evaluated
- If search backtracks, then all clauses of all newly unassigned variables are updated

Total number of references is $L$, where $L$ is the number of literals
Lazy Data Structures (watched literals)

- For each clause, only two variables keep a reference to the clause, i.e. only 2 literals are watched
  - If variable $x$ is assigned, only the clauses where literals in $x$ are watched need to be evaluated
  - If search backtracks, then nothing needs to be done
- Total number of references is $2 \times C$, where $C$ is the number of clauses
  - In general $L \gg 2 \times C$, in particular if clauses are learnt
Search Heuristics

- **Standard data structures:** heavy heuristics
  - DLIS: Dynamic Large Individual Sum [Marques-Silva’99]
    - Selects the literal that appears most frequently in unresolved clauses

- **Lazy data structures:** light heuristics
  - VSIDS: Variable State Independent Decaying Sum [Moskewicz&al’01]
    - Each literal has a counter, initialized to zero
    - When a new clause is recorded, the counter associated with each literal in the clause is incremented
    - The unassigned literal with the highest counter is chosen at each decision
  - Other variations
    - Counters updated also for literals in the clauses involved in conflicts [Goldberg&Novikov’02]
Restarts I

- Plot for processor verification instance with branching randomization and 10000 runs
  - More than 50% of the runs require less than 1000 backtracks
  - A small percentage requires more than 10000 backtracks

- Run times of backtrack search SAT solvers characterized by heavy-tail distributions
Repeatedly restart the search each time a cutoff is reached
- Randomization allows to explore different paths in search tree
- Resulting algorithm is incomplete
  - Increase the cutoff value
  - Keep clauses from previous runs
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Extensions
Well-Known Extensions of SAT

- The formula is unsatisfiable
  - Maximum Satisfiability (MAX-SAT):
    Satisfy the largest number of clauses
- Quantify the variables
  - Quantified Boolean Formulas (QBF):
    Boolean formulas where variables are existentially or universally quantified
- Consider extended constraints
  - Pseudo-Boolean formulas (PBS/PBO):
    Linear inequalities over Boolean variables
- Consider decidable fragments of FOL
  - Satisfiability Modulo Theories (SMT):
    Decision procedures for a number of theories exist
    - Linear Integer Arithmetic
    - Uninterpreted Functions
    - ...
- Interesting results for most extensions, but still far from the impact of SAT solvers
Conclusions

- The **ingredients** for having an efficient SAT solver
  - **Mistakes are not a problem**
    - Learn from your conflicts
    - ... and perform non-chronological backtracking
    - Restart the search
  - **Be lazy!**
    - Lazy data structures
    - Low-cost heuristics

- Thanks to João Marques-Silva and Daniel Le Berre
The Next SAT Conference

May 12 - 15 2008, Guangzhou, P. R. China
Submission deadline: January 11th, 2008
Affiliated events
- SAT Race
- QBFEVAL
- Max-SAT Evaluation
Thank you!