Exploiting Cardinality Encodings in Parallel Maximum Satisfiability

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What is Maximum Satisfiability?

CNF Formula:

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\neg x_4 \vee x_5$
$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

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- Formula is unsatisfiable

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$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

- Formula is unsatisfiable
- Maximum Satisfiability (MaxSAT):
  - Find an assignment that maximizes (minimizes) number of satisfied (unsatisfied) clauses.

Motivation

- MaxSAT has several applications:
  - Software package upgradability;
  - Bug localization in C code;
  - Design debugging;
  - ...
- Improving MaxSAT solvers will have a practical impact;
- Multicore processors are now predominant;
- Parallel solving is known to boost the performance of the solver;

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- Improving MaxSAT solvers will have a practical impact;
- Multicore processors are now predominant;
- Parallel solving is known to boost the performance of the solver;
- **Goal:** develop a new parallel MaxSAT solver;
- **How:** using a portfolio of algorithms.

Algorithms for solving MaxSAT

Unsatisfiability-based algorithms:

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\neg x_4 \vee x_5$
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Algorithms for solving MaxSAT

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$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

Formula is UNSAT; Get unsatisfiable sub-formula;
Lower bound value: 1

Algorithms for solving MaxSAT

Unsatisfiability-based algorithms:

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1 \vee r_1$	$\neg x_1 \vee r_2$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4 \vee r_3$	$\neg x_4 \vee x_5 \vee r_4$
$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3 \vee r_5$	$\neg x_3 \vee r_6$

$\sum_{i=1}^{6} r_i \leq 1$

Add relaxation variables and at-most-one constraint

Algorithms for solving MaxSAT

Unsatisfiability-based algorithms:

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1 \vee r_1$	$\neg x_1 \vee r_2$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4 \vee r_3$	$\neg x_4 \vee x_5 \vee r_4$
$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3 \vee r_5$	$\neg x_3 \vee r_6$
$\sum_{i=1}^{6} r_i \leq 1$			

Formula is UNSAT; Get unsatisfiable sub-formula;
Lower bound value: 2

Algorithms for solving MaxSAT

Unsatisfiability-based algorithms:

$x_6 \vee x_2 \vee r_7$ | $\neg x_6 \vee x_2 \vee r_8$ | $\neg x_2 \vee x_1 \vee r_1 \vee r_9$ | $\neg x_1 \vee r_2 \vee r_{10}$

$\neg x_6 \vee x_8$ | $x_6 \vee \neg x_8$ | $x_2 \vee x_4 \vee r_3$ | $\neg x_4 \vee x_5 \vee r_4$

$x_7 \vee x_5 \vee r_{11}$ | $\neg x_7 \vee x_5 \vee r_{12}$ | $\neg x_5 \vee x_3 \vee r_5 \vee r_{13}$ | $\neg x_3 \vee r_6 \vee r_{14}$

$\sum_{i=1}^{6} r_i \leq 1$ | $\sum_{i=7}^{14} r_i \leq 1$

Add relaxation variables and at-most-one constraint

Algorithms for solving MaxSAT

Unsatisfiability-based algorithms:

$x_6 \vee x_2 \vee r_7$	$\neg x_6 \vee x_2 \vee r_8$	$\neg x_2 \vee x_1 \vee r_1 \vee r_9$	$\neg x_1 \vee r_2 \vee r_{10}$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4 \vee r_3$	$\neg x_4 \vee x_5 \vee r_4$
$x_7 \vee x_5 \vee r_{11}$	$\neg x_7 \vee x_5 \vee r_{12}$	$\neg x_5 \vee x_3 \vee r_5 \vee r_{13}$	$\neg x_3 \vee r_6 \vee r_{14}$
$\sum_{i=1}^{6} r_i \leq 1$	$\sum_{i=7}^{14} r_i \leq 1$		

Formula is now SAT
The minimum number of unsatisfiable clauses is 2

Algorithms for solving MaxSAT

Linear search algorithms on the number of unsatisfiable clauses:

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\neg x_4 \vee x_5$
$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

Algorithms for solving MaxSAT

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$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

Formula is UNSAT

Algorithms for solving MaxSAT

Linear search algorithms on the number of unsatisfiable clauses:

$x_6 \vee x_2 \vee r_1$	$\neg x_6 \vee x_2 \vee r_2$	$\neg x_2 \vee x_1 \vee r_3$	$\neg x_1 \vee r_4$
$\neg x_6 \vee x_8 \vee r_5$	$x_6 \vee \neg x_8 \vee r_6$	$x_2 \vee x_4 \vee r_7$	$\neg x_4 \vee x_5 \vee r_8$
$x_7 \vee x_5 \vee r_9$	$\neg x_7 \vee x_5 \vee r_{10}$	$\neg x_5 \vee x_3 \vee r_{11}$	$\neg x_3 \vee r_{12}$

$\sum_{i=1}^{12} r_i \leq 12$

Add relaxation variables and at-most-k constraint

Algorithms for solving MaxSAT

Linear search algorithms on the number of unsatisfiable clauses:

$x_6 \vee x_2 \vee r_1$	$\neg x_6 \vee x_2 \vee r_2$	$\neg x_2 \vee x_1 \vee r_3$	$\neg x_1 \vee r_4$
$\neg x_6 \vee x_8 \vee r_5$	$x_6 \vee \neg x_8 \vee r_6$	$x_2 \vee x_4 \vee r_7$	$\neg x_4 \vee x_5 \vee r_8$
$x_7 \vee x_5 \vee r_9$	$\neg x_7 \vee x_5 \vee r_{10}$	$\neg x_5 \vee x_3 \vee r_{11}$	$\neg x_3 \vee r_{12}$
$\sum_{i=1}^{12} r_i \leq 12$			

Formula is SAT; solution found has $\sum_{i=1}^{12} r_i = 9$;
Upper bound value: 9

Algorithms for solving MaxSAT

Linear search algorithms on the number of unsatisfiable clauses:

$x_6 \vee x_2 \vee r_1$	$\neg x_6 \vee x_2 \vee r_2$	$\neg x_2 \vee x_1 \vee r_3$	$\neg x_1 \vee r_4$
$\neg x_6 \vee x_8 \vee r_5$	$x_6 \vee \neg x_8 \vee r_6$	$x_2 \vee x_4 \vee r_7$	$\neg x_4 \vee x_5 \vee r_8$
$x_7 \vee x_5 \vee r_9$	$\neg x_7 \vee x_5 \vee r_{10}$	$\neg x_5 \vee x_3 \vee r_{11}$	$\neg x_3 \vee r_{12}$

$\sum_{i=1}^{12} r_i \leq 8$

Update at-most-k constraint

Algorithms for solving MaxSAT

Linear search algorithms on the number of unsatisfiable clauses:

$x_6 \vee x_2 \vee r_1$	$\neg x_6 \vee x_2 \vee r_2$	$\neg x_2 \vee x_1 \vee r_3$	$\neg x_1 \vee r_4$
$\neg x_6 \vee x_8 \vee r_5$	$x_6 \vee \neg x_8 \vee r_6$	$x_2 \vee x_4 \vee r_7$	$\neg x_4 \vee x_5 \vee r_8$
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$\sum_{i=1}^{12} r_i \leq 8$			

Formula is SAT; solution found has $\sum_{i=1}^{12} r_i = 2$;
Upper bound value: 2

Algorithms for solving MaxSAT

Linear search algorithms on the number of unsatisfiable clauses:

$$x_6 \vee x_2 \vee r_1 \qquad \neg x_6 \vee x_2 \vee r_2 \qquad \neg x_2 \vee x_1 \vee r_3 \qquad \neg x_1 \vee r_4$$

$$\neg x_6 \vee x_8 \vee r_5 \qquad x_6 \vee \neg x_8 \vee r_6 \qquad x_2 \vee x_4 \vee r_7 \qquad \neg x_4 \vee x_5 \vee r_8$$

$$x_7 \vee x_5 \vee r_9 \qquad \neg x_7 \vee x_5 \vee r_{10} \qquad \neg x_5 \vee x_3 \vee r_{11} \qquad \neg x_3 \vee r_{12}$$

$\sum_{i=1}^{12} r_i \leq 1$

Update at-most-k constraint

Algorithms for solving MaxSAT

Linear search algorithms on the number of unsatisfiable clauses:

$x_6 \vee x_2 \vee r_1$	$\neg x_6 \vee x_2 \vee r_2$	$\neg x_2 \vee x_1 \vee r_3$	$\neg x_1 \vee r_4$
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$\sum_{i=1}^{12} r_i \leq 1$			

Formula is UNSAT;
The minimum number of unsatisfiable clauses is 2

Cardinality constraints in MaxSAT solving

- MaxSAT algorithms use cardinality constraints:
  - at-most-one;
  - at-most-k.
- Cardinality constraints:
  - $\sum_{i=1}^{n} l_i \leq k$, where $l_i \in \{x_i, \neg x_i\}$
- Handling cardinality constraints:
  - Encode cardinality constraints into clauses;
  - Use a native representation of cardinality constraints.

Encodings for cardinality constraints

- Cardinality constraints:
  - At-most-one
  - At-most-k

Encodings for cardinality constraints

- Cardinality constraints:
  - At-most-one
  - At-most-k

- Naive (pairwise) encoding for at-most-one constraints:
  - Cardinality constraint: $x_1 + x_2 + x_3 + x_4 \leq 1$
  - Clauses:

$$\left.\begin{array}{c} (x_1 \Rightarrow \neg x_2) \\ (x_1 \Rightarrow \neg x_3) \\ (x_1 \Rightarrow \neg x_4) \\ \dots \end{array}\right\} \begin{array}{c} \neg x_1 \vee \neg x_2 \\ \neg x_1 \vee \neg x_3 \\ \neg x_1 \vee \neg x_4 \\ \dots \end{array}$$

  - Complexity: $\mathcal{O}(n^2)$ clauses

Encodings for cardinality constraints

- Cardinality constraints:
  - At-most-one
  - At-most-k
- Naive encoding for at-most-k constraints:
  - Cardinality constraint: $x_1 + x_2 + x_3 + x_4 \leq 2$
  - Clauses:

$$\left.\begin{array}{c} (x_1 \wedge x_2 \Rightarrow \neg x_3) \\ (x_1 \wedge x_2 \Rightarrow \neg x_4) \\ (x_2 \wedge x_3 \Rightarrow \neg x_4) \\ \ldots \end{array}\right\} \begin{array}{c} (\neg x_1 \vee \neg x_2 \vee \neg x_3) \\ (\neg x_1 \vee \neg x_2 \vee \neg x_4) \\ (\neg x_2 \vee \neg x_3 \vee \neg x_4) \\ \ldots \end{array}$$

  - Complexity: $\mathcal{O}(n^k)$ clauses

Encodings for cardinality constraints

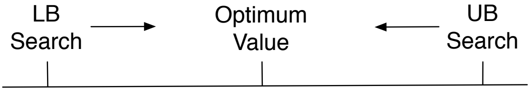
Encoding	Clauses	Variables	Type
Pairwise	$\mathcal{O}(n^2)$	0	at-most-one
Ladder	$\mathcal{O}(n)$	$\mathcal{O}(n)$	at-most-one
Bitwise	$\mathcal{O}(n\ log_2\ n)$	$\mathcal{O}(log_2\ n)$	at-most-one
Commander	$\mathcal{O}(n)$	$\mathcal{O}(n)$	at-most-one
Product	$\mathcal{O}(n)$	$\mathcal{O}(n)$	at-most-one
Sequential	$\mathcal{O}(nk)$	$\mathcal{O}(nk)$	at-most-k
Totalizer	$\mathcal{O}(nk)$	$\mathcal{O}(n\ log_2\ n)$	at-most-k
Sorters	$\mathcal{O}(n\ log_2^2\ n)$	$\mathcal{O}(n\ log_2^2\ n)$	at-most-k

Encodings for cardinality constraints

- Cardinality constraints:
  - At-most-one;
  - At-most-k.
- Many encodings for cardinality constraints are available;
- Different encodings can perform better over different problems;
- Exploit the diversification of cardinality encodings.

Parallel MaxSAT solver (2 threads)

- Search in the lower and upper bound values of the optimal solution:



- The optimum value is found when:
  - LB or UB search terminates with a solution;
  - or when LB value = UB value.

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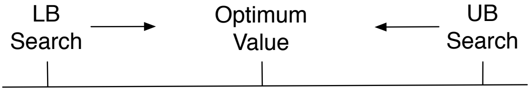

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T_0												T_1
LB												UB
0	1	2	3	4	5	6	7	8	9	10	11	12

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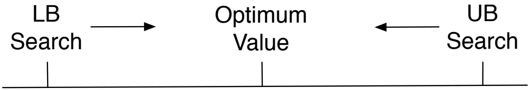

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T_0 returns UNSAT; update lower bound value

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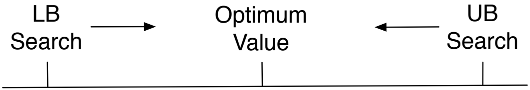

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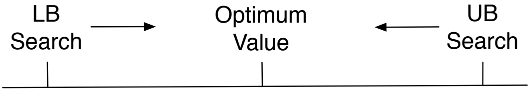
	T_0								T_1			
	LB								UB			
0	1	2	3	4	5	6	7	8	9	10	11	12

T_1 returns SAT; update upper bound value

Parallel MaxSAT solver (2 threads)

- Search in the lower and upper bound values of the optimal solution:



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		T_0							T_1			
		LB							UB			
0	1	2	3	4	5	6	7	8	9	10	11	12

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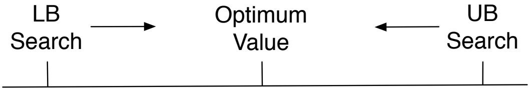

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		T_0; T_1										
		LB; UB										
0	1	2	3	4	5	6	7	8	9	10	11	12

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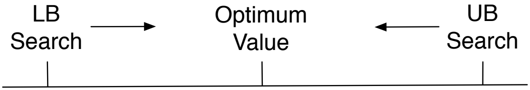

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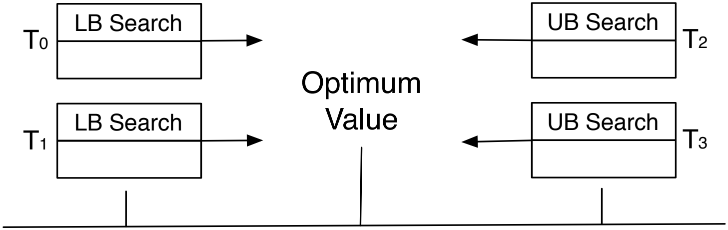
		T_0; T_1										
		LB; UB										
0	1	2	3	4	5	6	7	8	9	10	11	12

LB value = UB value, optimal value has been found

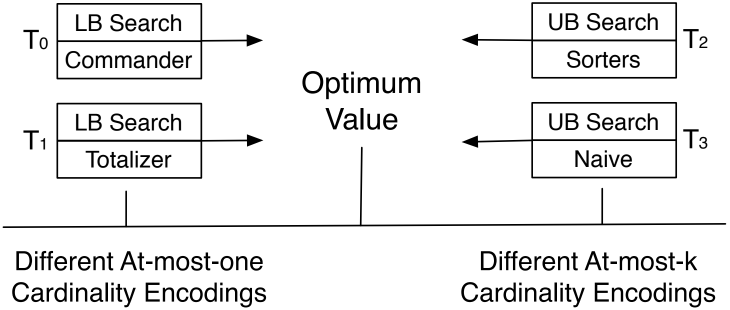
Parallel MaxSAT solver (n threads)

- Search in the lower and upper bound values of the optimal solution:



Parallel MaxSAT solver (*n* threads)

- Search in the lower and upper bound values of the optimal solution:



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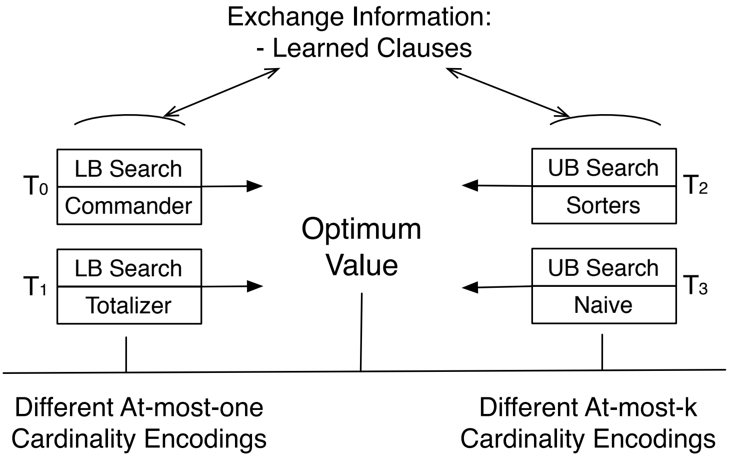


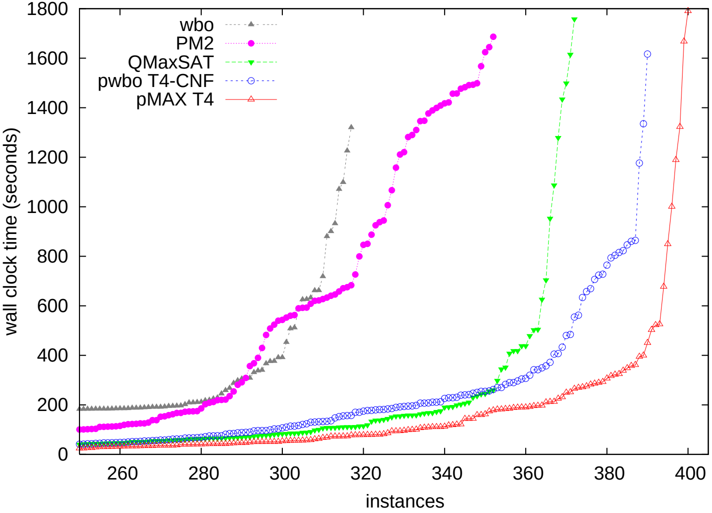
Experimental Results

- Sequential Solvers:
  - QMaxSAT:
    - Upper bound search solver;
    - Uses cardinality encodings.
  - PM2:
    - Lower bound search solver;
    - Uses cardinality encodings.
  - wbo:
    - Upper bound search during 10% of the time limit;
    - Lower bound search on remaining time;
    - Does not use cardinality encodings.

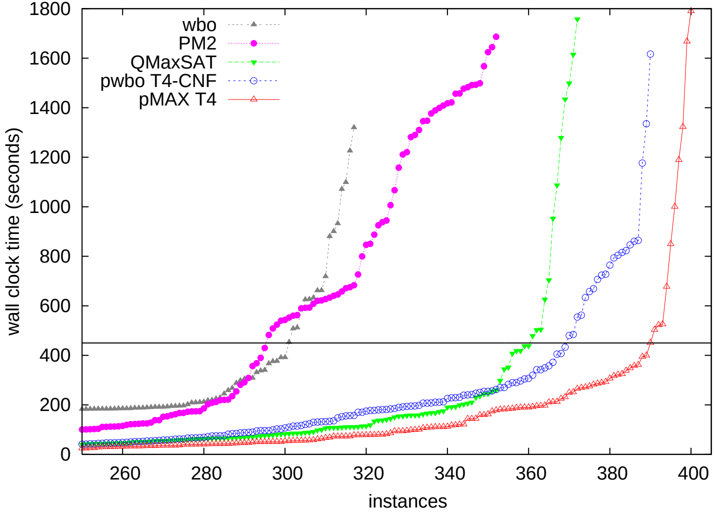
Experimental Results

- Parallel Solvers:
  - pwbo-CNF:
    - Searches on the lower and upper bound values of the optimal solution;
    - Splits the search on different upper bound values;
    - Only one thread uses cardinality encodings.
  - pMAX:
    - The solver proposed in this talk;
    - Searches on the lower and upper bound values of the optimal solution;
    - Uses a different cardinality encoding in each thread.

Experimental Results

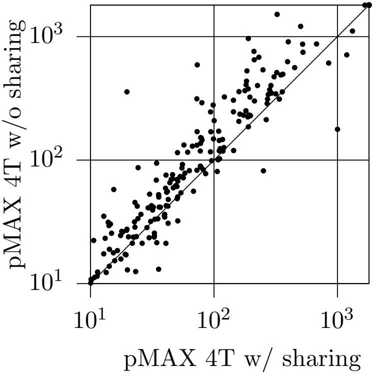


Experimental Results



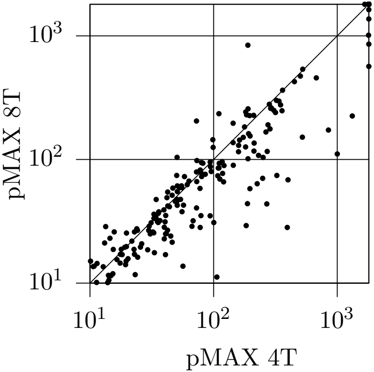
Experimental Results

- Impact of sharing learned clauses (seconds):



Experimental Results

- Scalability of pMAX (seconds):



Experimental Results

- Speedup on instances solved by all our solvers:

Solver	Time (s)	Speedup
wbo	67,947.41	1.00
pwbo 4T-CNF	18,015.69	3.77
pMAX 4T	11,382.91	5.97
pMAX 8T	7,990.10	8.50

Conclusions

- Diversification of cardinality encodings can be used in parallel MaxSAT;
- pMAX outperforms state-of-the-art MaxSAT solvers:
  - Even when considering CPU time.
- Sharing learned clauses has a strong impact on the solving speed;
- pMAX shows scalability:
  - pMAX 8T is 1.4$\times$ faster than pMAX 4T.