Clause Sharing in Deterministic Parallel Maximum Satisfiability

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Maximum Satisfiability

- Maximum Satisfiability (MaxSAT):
  - Optimization version of Boolean Satisfiability (SAT);
  - **Goal:** Given a propositional formula $\varphi$, find an assignment to problem variables that maximizes (minimizes) number of satisfied (unsatisfied) clauses in $\varphi$. 

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- Partial MaxSAT
  - **Goal:** Given a propositional formula $\varphi = \varphi_h \cup \varphi_s$, find an assignment to problem variables such that all *hard* clauses in $\varphi_h$ are satisfied, while minimizing the number of unsatisfied *soft* clauses in $\varphi_s$. 
Maximum Satisfiability

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  - Optimization version of Boolean Satisfiability (SAT);
  - **Goal:** Given a propositional formula \( \varphi \), find an assignment to problem variables that maximizes (minimizes) number of satisfied (unsatisfied) clauses in \( \varphi \).

- **Partial MaxSAT**
  - **Goal:** Given a propositional formula \( \varphi = \varphi_h \cup \varphi_s \), find an assignment to problem variables such that all hard clauses in \( \varphi_h \) are satisfied, while minimizing the number of unsatisfied soft clauses in \( \varphi_s \).
Maximum Satisfiability

- Main algorithmic approaches:
  - Branch and Bound
    - Extensive use of lower bounding procedures
    - Restrictive use of MaxSAT inference rules
  - Linear search on the number of unsatisfied clauses
    - Each time a new restriction is found, a new constraint is added that excludes solutions with higher cost
  - Unsatisfiability-based solvers
    - Iterative identification of unsatisfiable subformulas

Our focus is on the latter two approaches since these have been shown to be more effective in Industrial instances
Linear search on the number of unsatisfied clauses

Best: $\infty$

$$x_6 \lor x_2 \quad \neg x_6 \lor x_2 \quad \neg x_2 \lor x_1 \quad \neg x_1$$

$$\neg x_6 \lor x_8 \quad x_6 \lor \neg x_8 \quad x_2 \lor x_4 \quad \neg x_4 \lor x_5$$

$$x_7 \lor x_5 \quad \neg x_7 \lor x_5 \quad \neg x_5 \lor x_3 \quad \neg x_3$$

Example of MaxSAT formula; Hard clauses in blue; Soft in red
Linear search on the number of unsatisfied clauses

Best: $\infty$

$x_6 \lor x_2$  $\neg x_6 \lor x_2$  $\neg x_2 \lor x_1$  $\neg x_1 \lor r_1$

$\neg x_6 \lor x_8 \lor r_2$  $x_6 \lor \neg x_8 \lor r_3$  $x_2 \lor x_4 \lor r_4$  $\neg x_4 \lor x_5 \lor r_5$

$x_7 \lor x_5$  $\neg x_7 \lor x_5 \lor r_6$  $\neg x_5 \lor x_3$  $\neg x_3 \lor r_7$

Add a relaxation variable to each soft clause; All clauses are now considered hard
Linear search on the number of unsatisfied clauses

Best: \( \infty \)

\[
x_6 \lor x_2 \\
x_6 \lor x_8 \lor r_2 \\
x_7 \lor x_5 \\
\neg x_6 \lor x_2 \\
x_6 \lor \neg x_8 \lor r_3 \\
x_7 \lor x_5 \lor r_6 \\
\neg x_2 \lor x_1 \\
x_2 \lor x_4 \lor r_4 \\
\neg x_5 \lor x_3 \\
\neg x_1 \lor r_1 \\
\neg x_4 \lor x_5 \lor r_5 \\
\neg x_3 \lor r_7
\]

Goal is to find an assignment that minimizes the number of relaxation variables assigned value 1
Linear search on the number of unsatisfied clauses

Best: 4

\[
\begin{align*}
\neg x_6 \lor x_2 \\
\neg x_6 \lor x_8 \lor r_2 \\
x_7 \lor x_5 \\
\neg x_7 \lor x_5 \lor r_6
\end{align*}
\]

Find a solution; Suppose a solution is found such that 4 relaxation variables are assigned value 1;
Linear search on the number of unsatisfied clauses

Best: 4

$x_6 \lor x_2$  $\neg x_6 \lor x_2$  $\neg x_2 \lor x_1$  $\neg x_1 \lor r_1$

$\neg x_6 \lor x_8 \lor r_2$  $x_6 \lor \neg x_8 \lor r_3$  $x_2 \lor x_4 \lor r_4$  $\neg x_4 \lor x_5 \lor r_5$

$x_7 \lor x_5$  $\neg x_7 \lor x_5 \lor r_6$  $\neg x_5 \lor x_3$  $\neg x_3 \lor r_7$

$\sum_{i=1}^{7} r_i \leq 3$

Add new constraint that excludes solutions with equal or higher cost;
Linear search on the number of unsatisfied clauses

Best: 2

\[ x_6 \lor x_2 \]  \[ \neg x_6 \lor x_2 \]  \[ \neg x_2 \lor x_1 \]  \[ \neg x_1 \lor r_1 \]

\[ \neg x_6 \lor x_8 \lor r_2 \]  \[ x_6 \lor \neg x_8 \lor r_3 \]  \[ x_2 \lor x_4 \lor r_4 \]  \[ \neg x_4 \lor x_5 \lor r_5 \]

\[ x_7 \lor x_5 \]  \[ \neg x_7 \lor x_5 \lor r_6 \]  \[ \neg x_5 \lor x_3 \]  \[ \neg x_3 \lor r_7 \]

\[ \sum_{i=1}^{7} r_i \leq 3 \]

Find another solution; Suppose a solution is found such that 2 relaxation variables are assigned value 1;
Linear search on the number of unsatisfied clauses

Best: 2

\[
x_6 \lor x_2 \quad \neg x_6 \lor x_2 \quad \neg x_2 \lor x_1 \quad \neg x_1 \lor r_1
\]

\[
\neg x_6 \lor x_8 \lor r_2 \quad x_6 \lor \neg x_8 \lor r_3 \quad x_2 \lor x_4 \lor r_4 \quad \neg x_4 \lor x_5 \lor r_5
\]

\[
x_7 \lor x_5 \quad \neg x_7 \lor x_5 \lor r_6 \quad \neg x_5 \lor x_3 \quad \neg x_3 \lor r_7
\]

\[
\sum_{i=1}^{7} r_i \leq 3 \quad \sum_{i=1}^{7} r_i \leq 1
\]

Add new constraint that excludes solutions with equal or higher cost;
Linear search on the number of unsatisfied clauses

Best: 2

$x_6 \lor x_2$ \quad \neg x_6 \lor x_2 \quad \neg x_2 \lor x_1 \quad \neg x_1 \lor r_1$

$\neg x_6 \lor x_8 \lor r_2$ \quad $x_6 \lor \neg x_8 \lor r_3$ \quad $x_2 \lor x_4 \lor r_4$ \quad $\neg x_4 \lor x_5 \lor r_5$

$x_7 \lor x_5$ \quad $\neg x_7 \lor x_5 \lor r_6$ \quad $\neg x_5 \lor x_3$ \quad $\neg x_3 \lor r_7$

$\sum_{i=1}^{7} r_i \leq 3$ \quad $\sum_{i=1}^{7} r_i \leq 1$

Instance is now UNSAT; Optimal solution is to have two unsatisfied soft clauses
Unsatisfiability-based MaxSAT solvers

Example of MaxSAT formula; Hard clauses in blue; Soft in red;
Unsatisfiability-based MaxSAT solvers

\[ x_6 \lor x_2 \]
\[ \neg x_6 \lor x_8 \]
\[ x_7 \lor x_5 \]

\[ \neg x_6 \lor x_2 \]
\[ x_6 \lor \neg x_8 \]
\[ \neg x_7 \lor x_5 \]

Formula is unsat; Get Unsatisfiable subformula (Unsat Core)
Unsatisfiability-based MaxSAT solvers

\[ x_6 \lor x_2 \quad \neg x_6 \lor x_2 \]
\[ \neg x_6 \lor x_8 \quad x_6 \lor \neg x_8 \]
\[ x_7 \lor x_5 \quad \neg x_7 \lor x_5 \]

\[ \sum_{i=1}^{4} r_i \leq 1 \]

Add relaxation variables to soft clauses and AtMost1 constraint
Unsatisfiability-based MaxSAT solvers

\[\begin{align*}
x_6 \lor x_2 & \quad \neg x_6 \lor x_2 & \quad \neg x_2 \lor x_1 & \quad \neg x_1 \lor r_1 \\
\neg x_6 \lor x_8 & \quad x_6 \lor \neg x_8 & \quad x_2 \lor x_4 \lor r_2 & \quad \neg x_4 \lor x_5 \lor r_3 \\
x_7 \lor x_5 & \quad \neg x_7 \lor x_5 & \quad \neg x_5 \lor x_3 & \quad \neg x_3 \lor r_4 \\
\sum_{i=1}^{4} r_i \leq 1
\end{align*}\]

Formula is still unsat; Get another Unsat Core
Unsatisfiability-based MaxSAT solvers

\[ x_6 \lor x_2 \quad \neg x_6 \lor x_2 \quad \neg x_2 \lor x_1 \quad \neg x_1 \lor r_1 \lor r_5 \]

\[ \neg x_6 \lor x_8 \quad x_6 \lor \neg x_8 \quad x_2 \lor x_4 \lor r_2 \quad \neg x_4 \lor x_5 \lor r_3 \]

\[ x_7 \lor x_5 \quad \neg x_7 \lor x_5 \lor r_6 \quad \neg x_5 \lor x_3 \quad \neg x_3 \lor r_4 \lor r_7 \]

\[ \sum_{i=1}^{4} r_i \leq 1 \quad \sum_{i=5}^{7} r_i \leq 1 \]

Add new relaxation variables to soft clauses in Unsat Core and AtMost1 constraint
Unsatisfiability-based MaxSAT solvers

\[
\begin{align*}
&x_6 \lor x_2 & \neg x_6 \lor x_2 & \neg x_2 \lor x_1 & \neg x_1 \lor r_1 \lor r_5 \\
&\neg x_6 \lor x_8 & x_6 \lor \neg x_8 & x_2 \lor x_4 \lor r_2 & \neg x_4 \lor x_5 \lor r_3 \\
&x_7 \lor x_5 & \neg x_7 \lor x_5 \lor r_6 & \neg x_5 \lor x_3 & \neg x_3 \lor r_4 \lor r_7
\end{align*}
\]

\[
\sum_{i=1}^{4} r_i \leq 1 \quad \sum_{i=5}^{7} r_i \leq 1
\]

Instance is now SAT; Algorithm Ends; Optimal solution is to have two unsatisfied soft clauses
Parallel MaxSAT Solvers

- **PWBO** is a parallel MaxSAT solver based on having several threads running a portfolio of two orthogonal algorithms:
  - an unsatisfiability-based algorithm that searches on the lower bound of the optimal solution;
  - a classical linear search algorithm that searches on the upper bound.
Parallel MaxSAT Solvers

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  - a classical linear search algorithm that searches on the upper bound.

![Diagram of Parallel MaxSAT Solvers](image)
Parallel MaxSAT Solvers

- Shared Clause: a clause that is *shared* by a thread to be used in other threads;
- Imported Clause: a clause that is *imported* by a thread;
Parallel MaxSAT Solvers

- Shared Clause: a clause that is *shared* by a thread to be used in other threads;
- Imported Clause: a clause that is *imported* by a thread;
- Not all learned clauses should be shared/imported since it could lead to an exponential blow up in memory;
- Shared clauses can be imported or discarded by the receiving thread;
Parallel MaxSAT Solvers

- **Shared Clause**: a clause that is *shared* by a thread to be used in other threads;
- **Imported Clause**: a clause that is *imported* by a thread;

- Not all learned clauses should be shared/imported since it could lead to an exponential blow up in memory;
- Shared clauses can be imported or discarded by the receiving thread;

**Question**: which learned clauses should be shared/imported by the different threads?
Clause Sharing Heuristics

- **Static:**
  - Learned clauses are shared/imported within a given cutoff.

- **Dynamic:**
  - Dynamic heuristics adjust the cutoff during the search.

- **Freezing:**
  - Shared clauses are temporarily frozen until they are expected to be useful.
Clause Sharing Heuristics (Static)

• **Size:**
  ○ The clause size is given by the number of literals;
  ○ Small clauses are expected to be more useful than larger clauses.

• **Literal Block Distance (LBD):**
  ○ The literal block distance corresponds to the number of different decision levels involved in a clause;
  ○ Clauses with small LBD are considered as more relevant.

• **Random:**
  ○ Randomly decide whether to share each learned clause with a given probability.
Clause Sharing Heuristics (Dynamic)

• The size of learned clauses tends to increase over time;
• Dynamic heuristics adjust the size of shared clauses during the search;

• Hamadi et al. proposed the following dynamic heuristic:
  ○ At every $k$ conflicts the throughput of shared clauses is evaluated between each pair of threads ($t_i \rightarrow t_j$);
  ○ If the sharing is small, the cutoff is dynamically increased;
  ○ If the sharing is large, the cutoff is dynamically reduced.
Clause Sharing Heuristics (Dynamic)

- The previous heuristic has been improved by Hamadi et al. by considering the quality of shared clauses:
  - A shared clause is said to have *quality* if at least half of its literals are active;
  - A literal is *active* if the variable’s decision heuristic score is high, i.e. it is likely to be chosen as a decision variable in the near future;
  - If the quality is high then the increase (decrease) in the size limit of shared clauses will be larger (smaller).

- The reasoning behind this heuristic is that the information recently received from a thread $t_i$ is qualitatively linked to the information which could be received from the same thread $t_i$ in the near future.
Clause Sharing Heuristics (Freezing)

Freezing procedure for importing clauses shared by other threads

1. Shared Clauses
2. For each clause $\omega$
3. Freeze($\omega$)?
   - Yes: Reevaluate Clauses
   - No: Import $\omega$

Cleaning
Clause Sharing Heuristics (Freezing)

The freezing heuristic:

- Considers the status of the shared clause $\omega$ in the context of the importing thread:
  - *Satisfied*: if at least one of its literals is satisfied;
  - *Unsatisfied*: if all of its literals are unsatisfied;
  - *Unit*: if all literals but one are unsatisfied and the remaining literal is unassigned;
  - *Unresolved*: if it is not satisfied, unsatisfied or unit.

- Freezes shared clauses $\omega$ that are not likely to be useful in the near future.
Clause Sharing Heuristics (Freezing)

• A satisfied clause is expected to be useful in the near future if:
  ○ It is not necessary to backtrack significantly to make the clause unit;
  ○ The number of unassigned literals that are not active literals is small;

• Unsatisfied clauses and unit clauses are always useful to the current search;

• An Unresolved clause is expected to be useful in the near future if:
  ○ The number of unassigned literals that are not active literals is small;
Clause Sharing Heuristics (Evaluation)

**Question:** How to properly evaluate all these clause sharing heuristics?
Clause Sharing Heuristics (Evaluation)

Question: How to properly evaluate all these clause sharing heuristics?

Observe that:

• Parallel solvers are non-deterministic due to cooperation between threads
• Cooperation is known to boost the performance of parallel solvers
• Variations might result from other factors than clause sharing procedures
• Therefore, a more stable environment is required for a fair evaluation
Clause Sharing Heuristics (Evaluation)

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Observe that:

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Proposed approach: test different clause sharing heuristics in a **deterministic** parallel MaxSAT solver
Deterministic Parallel MaxSAT Solver

- Cooperation between threads must be deterministic
- Introduction of synchronization points
- Information is only exchanged at synchronization points
- When a thread reaches a synchronization point, waits until all other threads reach the same point
- Only when all threads stop at the synchronization point the information exchange takes place
Deterministic Parallel MaxSAT Solver

Thread$_1^{lb}$
- LB Search (export: clauses)
- export: core
- import: core, clauses
- LB Search (export: clauses)
- ... 
- end

Thread$_2^{lb}$
- LB Search (export: clauses)
- export: core
- import: core, clauses
- LB Search (export: clauses)
- ... 
- end

Thread$_3^{ub}$
- UB Search (export: clauses)
- export: solution, UB value
- import: UB value, clauses
- UB Search (export: clauses)
- optimal solution export: solution
- end

Thread$_4^{ub}$
- UB Search (export: clauses)
- export: solution, UB value
- import: UB value, clauses
- UB Search (export: clauses)
- ... 
- end
The definition of synchronization points must be deterministic.

Example: Synchronize after \( k \) conflicts.

If \( k \) is small, the number of synchronization points is large and threads are idle more often.

If \( k \) is large, there is little cooperation between threads.

For our experiments, we defined \( k = 100 \).

New ways of defining synchronization points are being tested.
Experimental Results

- Benchmarks: partial MaxSAT instances from the industrial category of the MaxSAT Evaluation 2011:
  - Instances that took less than 60 seconds to be solved were not considered;
- AMD Opteron 6172 processors (2.1 GHz with 64 GB of RAM) running Fedora Core 13;
- Timeout: 1,800 seconds (wall clock time);
- Portfolio version of \texttt{PWBO} with 4 threads:
  - A deterministic version of \texttt{PWBO} was used;
  - Information is only exchanged at synchronization points (every 100 conflicts).
Experimental Results

Comparison of the different heuristics for sharing learned clauses

<table>
<thead>
<tr>
<th>Heuristic</th>
<th>#Solved</th>
<th>Avg. #Clauses</th>
<th>Avg. Size</th>
<th>Time</th>
<th>Speedup</th>
</tr>
</thead>
<tbody>
<tr>
<td>No sharing</td>
<td>137</td>
<td>—</td>
<td>—</td>
<td>32,188.57</td>
<td>1.00</td>
</tr>
<tr>
<td>Static</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Random 30</td>
<td>134</td>
<td>10,140.22</td>
<td>128.21</td>
<td>27,394.46</td>
<td>1.18</td>
</tr>
<tr>
<td>LBD 5</td>
<td>137</td>
<td>8,947.36</td>
<td>9.94</td>
<td>25,346.69</td>
<td>1.27</td>
</tr>
<tr>
<td>Size 8</td>
<td>137</td>
<td>7,529.18</td>
<td>5.30</td>
<td>25,098.85</td>
<td>1.28</td>
</tr>
<tr>
<td>Size 32</td>
<td>138</td>
<td>18,027.48</td>
<td>11.76</td>
<td>25,174.29</td>
<td>1.28</td>
</tr>
<tr>
<td>Dynamic</td>
<td>138</td>
<td>13,296.28</td>
<td>7.33</td>
<td>24,218.84</td>
<td>1.33</td>
</tr>
<tr>
<td>Freezing</td>
<td>140</td>
<td>16,228.53</td>
<td>11.01</td>
<td>21,611.21</td>
<td>1.49</td>
</tr>
</tbody>
</table>

- Randomly sharing clauses deteriorates the performance;
- LBD and size heuristics have similar speedups;
- Dynamic heuristic outperforms the static heuristics but is outperformed by the freezing heuristic.
Experimental Results

Non-deterministic vs. Deterministic version

<table>
<thead>
<tr>
<th>Solver</th>
<th>#Solved</th>
<th>Time (s)</th>
<th>Avg. Idle CPU (%)</th>
<th>Speedup</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-Deterministic</td>
<td>141</td>
<td>13,401.88</td>
<td>0</td>
<td>1.00</td>
</tr>
<tr>
<td>Deterministic</td>
<td>140</td>
<td>21,611.21</td>
<td>43.12</td>
<td>0.62</td>
</tr>
</tbody>
</table>

• Deterministic version is slower
• Number of solved instances is very similar
• Large idle times should be decreased with other synchronization techniques
Conclusions

• Parallel MaxSAT solvers are now emerging:
  ◦ Sharing learned clauses boosts the performance of the solver.

• Heuristics are used for sharing learned clauses:
  ◦ Static, Dynamic and Freezing.

• Impact of sharing learned clauses in parallel MaxSAT:
  ◦ Number of solved instances does not increase significantly;
  ◦ Solving time is considerably reduced.

• The freezing heuristic outperforms all other heuristics both in solving time and number of solved instances.

• Deterministic parallel MaxSAT solver is slower but is still able to solve almost all instances solved by the non-deterministic version