Machine learning of strategies for efficiently solving QBF with abstraction refinement

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Quantified Boolean Formulae

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$\forall x.(x \leftrightarrow 0) \vee (x \leftrightarrow 1)$
$((0 \leftrightarrow 0) \vee (0 \leftrightarrow 1)) \wedge ((1 \leftrightarrow 0) \vee (1 \leftrightarrow 1))$
1

QBF is the paradigmatic PSPACE-complete problem

- Model checking
- Circuit synthesis
- Non-monotonic reasoning
- Conformant planning
- ...

Games and strategies

- We consider prenex form: *Quantifier-prefix*. *Matrix*
  Example $\forall x_1 x_2 \exists y_1 y_2 .\, (\neg x_1 \vee y_1) \wedge (x_2 \vee \neg y_2)$

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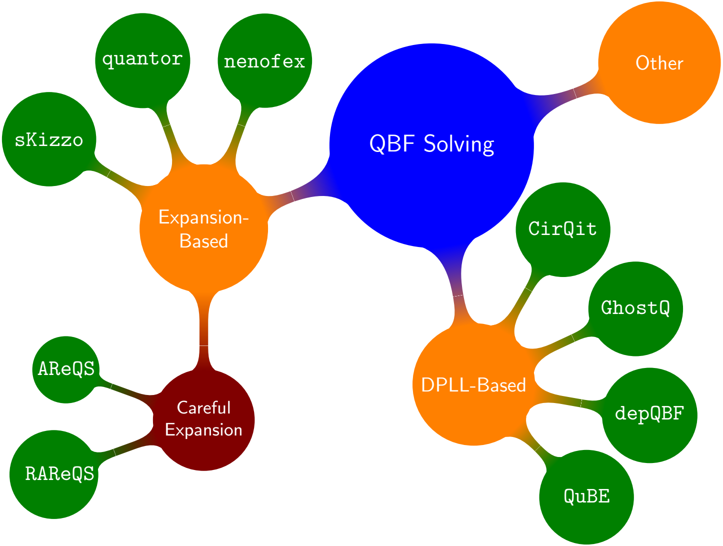
- **Example**

$$\forall u \exists e.\, u \leftrightarrow e$$

Solving QBF: DPLL versus expansion based solvers



Solving by Expansion

$$\exists x_1 \ldots x_m \forall y_1 \ldots y_m.\ \phi$$

Equivalent to:

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Equivalent to:

$$(\exists x_1 \ldots x_m) \left.\begin{array}{cccccc} & & \boldsymbol{y_1}, & \ldots, & \boldsymbol{y_{n-1}}, & \boldsymbol{y_n} \\ \wedge & \phi[& 0, & \ldots, & 0, & 0 \quad] \\ \wedge & \phi[& 0, & \ldots, & 0, & 1 \quad] \\ \wedge & \phi[& 0, & \ldots, & 1, & 0 \quad] \\ \wedge & \phi[& 0, & \ldots, & 1, & 1 \quad] \\ \wedge & \ldots & & & & \\ \wedge & \phi[& 1, & \ldots, & 1, & 1 \quad] \end{array}\right\}$$

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\end{array}
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$$

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- **Example:**

$$\exists x_1 x_2 \forall y_1 y_2 .\ (x_1 \wedge x_2 \wedge y_1 \wedge y_2)$$

- . . . sufficient to expand $y_1 = y_2 = 0$

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- **Example:**

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- ... sufficient to expand $y_1 = y_2 = 0$
- How to come up with the right expansions?

CEGAR paradigm : careful expansion

- Expand the formula gradually, to avoid exponential blow-up.
- . . . means gradually strengthening abstraction of the formula.

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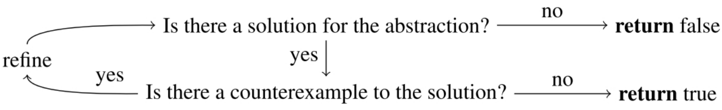

CEGAR loop

refine = expand more

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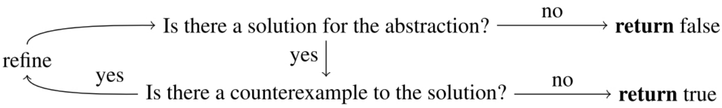

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AReQS: CEGAR-based solver for 2-QBF [J. and Silva SAT'11]

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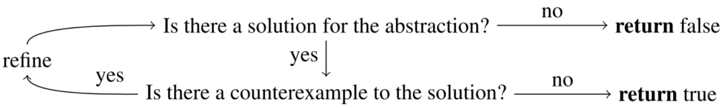

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RAReQS: generalises AReQS through recursion **[J. et al. SAT'12]**

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Move	Counter-move
$000\ldots0001$	

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000 . . . 0001	000 . . . 0001
000 . . . 0010	

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$000\ldots0010$	$000\ldots0010$

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Expansion necessarily exponential

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100 . . . 1000	000 . . . 1000
010 . . . 0011	010 . . . 0011
⋮	

$y_i \triangleq x_i$ is **learnt from** $\ll 2^n$ **expansions**

[J. AAAI'18]

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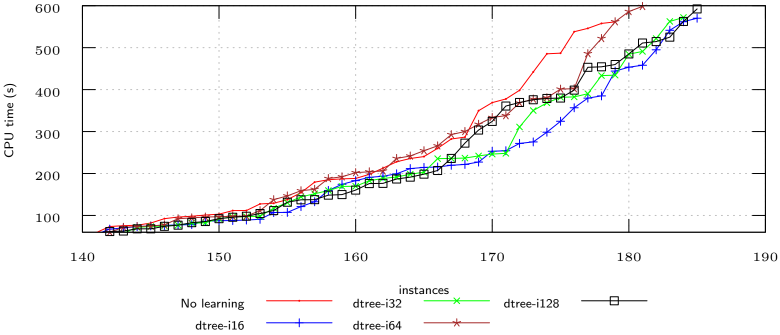
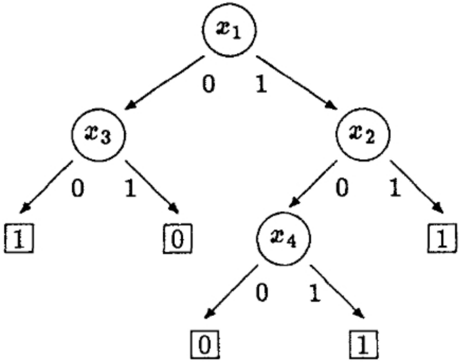
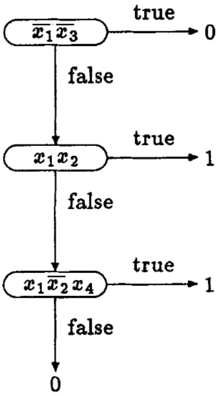
- Our samples are small (especially for ML standards), but the number of variables can be quite large.

Originally: Decision Trees and ID3



Alternative to Decision Trees: Decision Lists





Learning Algorithms

- k-decision list ... each rule at most k literals
- k-decision list are PAC-learnable **[Rivest '87]**

TÉCNICO LISBOA

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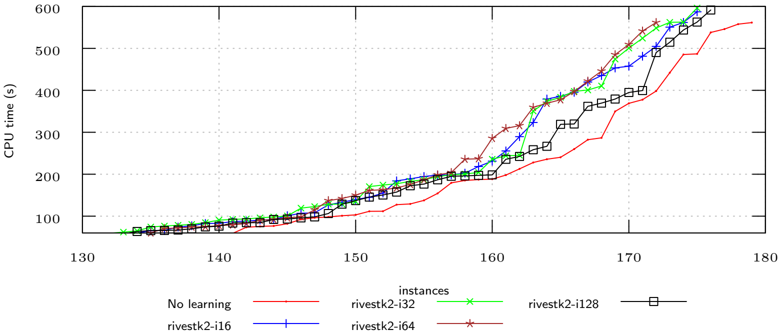
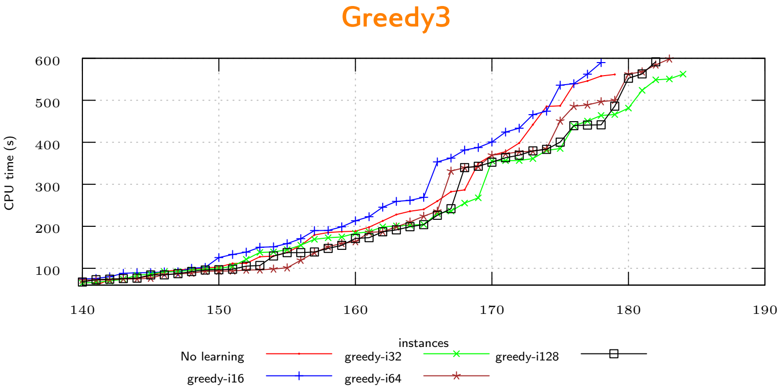
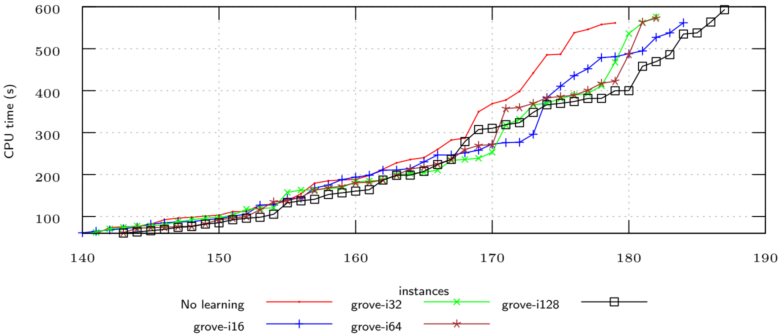
Decision Lists and Rivest ($k = 2$)




Greedy3
600
500
400
300
200
100
CPU time (s)
140
150
160
170
180
190
instances
No learning
greedy-i32
greedy-i128
greedy-i16
greedy-i64

Grove


CPU time (s)
600
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instances
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grove-i32
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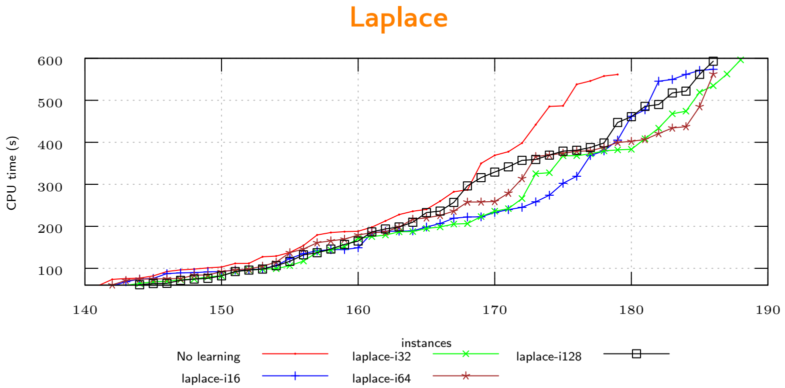
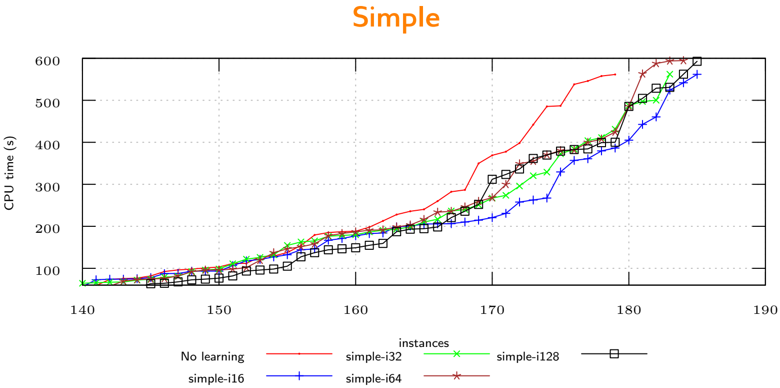

Laplace
CPU time (s)
600
500
400
300
200
100
140
150
160
170
180
190
instances
No learning
laplace-i16
laplace-i32
laplace-i64
laplace-i128


Simple
600
500
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CPU time (s)
140
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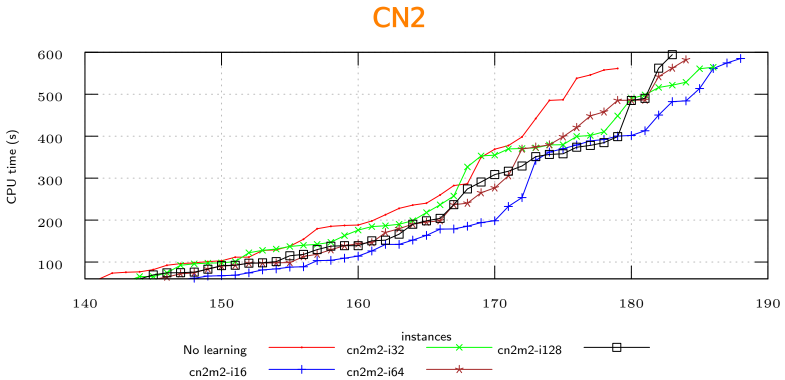
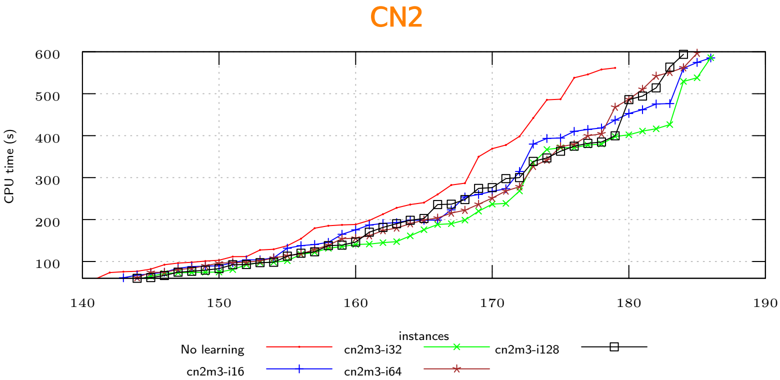
190
instances
No learning
simple-i32
simple-i128
simple-i16
simple-i64


CN2
600
500
400
300
200
100
CPU time (s)
140
150
160
170
180
190
instances
No learning
cn2m2-i32
cn2m2-i128
cn2m2-i16
cn2m2-i64


CN2
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600
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190
instances
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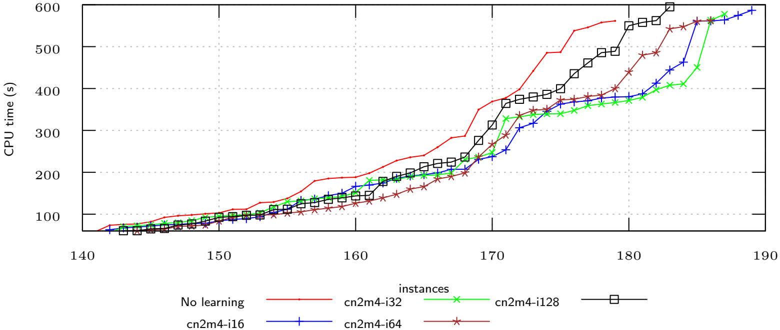
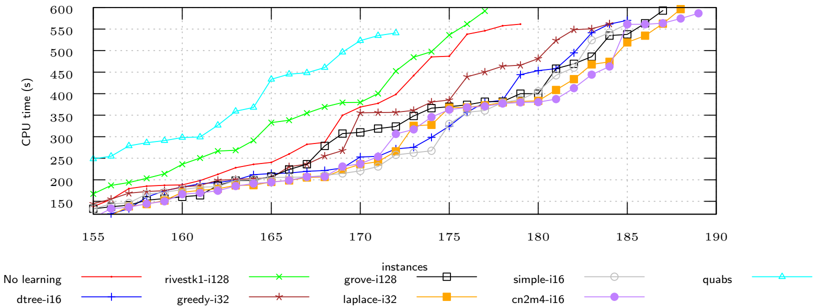
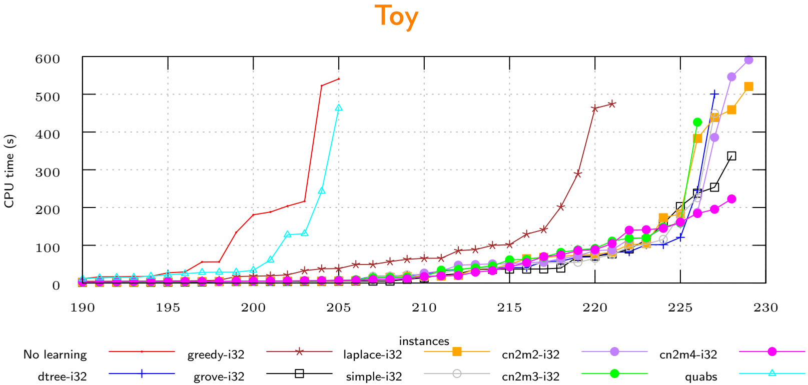
cn2m3-i32
cn2m3-i128
cn2m3-i16
cn2m3-i64

CN2

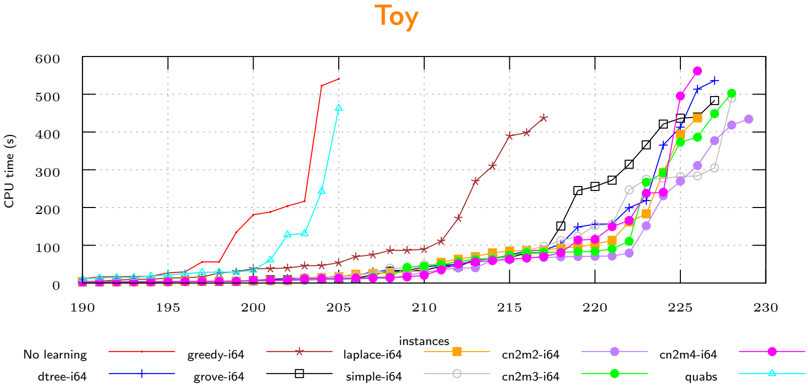

600
500
400
300
200
100
CPU time (s)
140
150
160
170
180
190
instances
No learning
cn2m4-i32
cn2m4-i128
cn2m4-i16
cn2m4-i64

Overview


600
550
500
450
400
350
300
250
200
150
CPU time (s)
155
160
165
170
175
180
185
190
instances
No learning
rivestk1-i128
grove-i128
simple-i16
quabs
dtree-i16
greedy-i32
laplace-i32
cn2m4-i16



270 instances, consisting of QBF encodings for a number of basic building blocks of circuits



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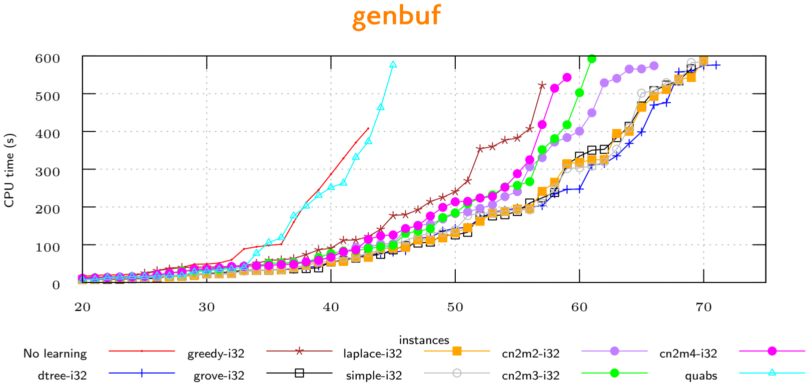
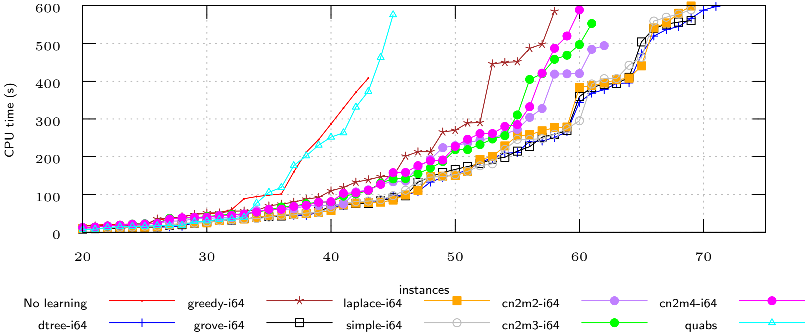
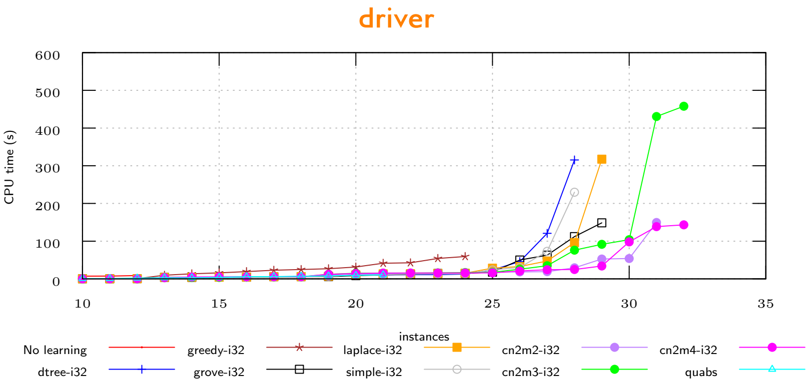


126 instances of encodings for generalized buffer specification

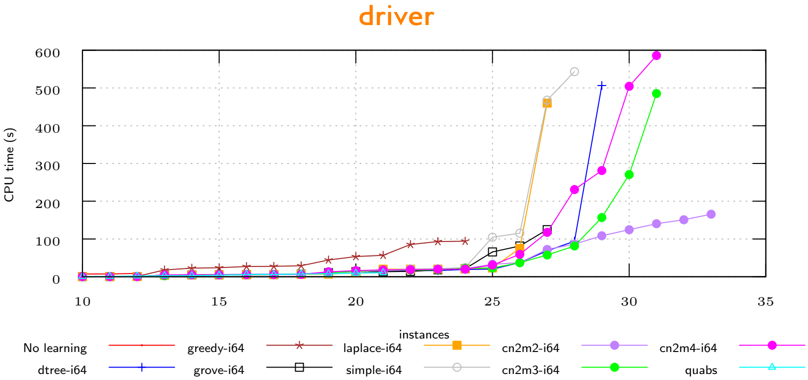
genbuf



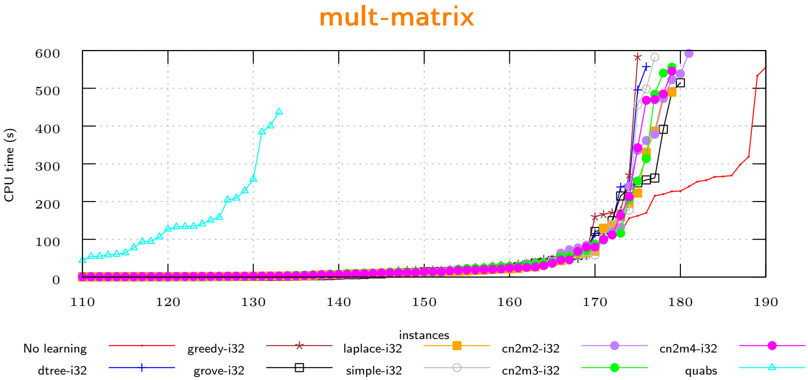
126 instances of encodings for generalized buffer specification



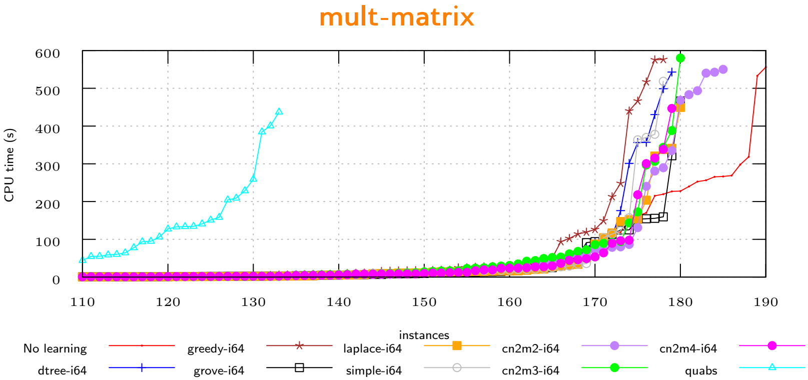
48 instances encoding specifications of a driver for a hard disk controller



48 instances encoding specifications of a driver for a hard disk controller



522 QBFs encoding the specifications for circuits that perform a single matrix multiplication, or repeated multiplication with a subset of controllable inputs



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Conclusions and Future Work

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- For some families of QBF, QFUN with learning is particularly useful, namely the families **toy**, **genbuf**, **driver** and **cycle-sched**.

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- Parallelism / solver portfolio