ON Q-RESOLUTION AND CDCL QBF SOLVING

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long-distance Q-resolution [Balabanov and Jiang, 2012] enables tautologous resolvents in *some cases*.

Q-RESOLUTION

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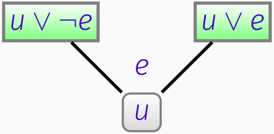
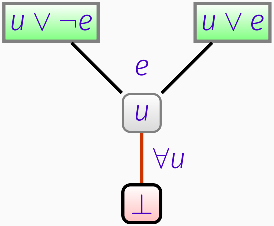


$$\forall u \exists e.\ (u \lor \neg e) \land (u \lor e)$$



$$\forall u \exists x.\, (u \vee \neg x) \wedge (\neg u \vee x)$$

$\neg u \vee x$ $\quad$ $u \vee \neg x$

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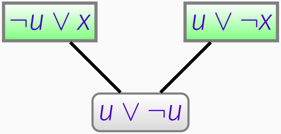

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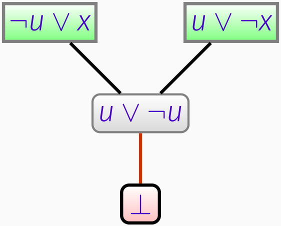

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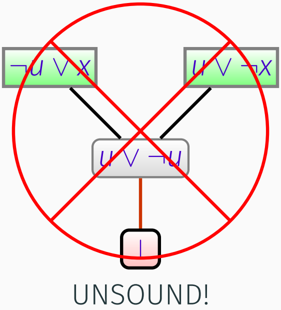

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- Propagation + UIP Learning via Q-Resolution

- Easy to defeat [Lonsing, 2012]:

$$\exists x_1 \ldots x_n \forall z \exists y_1, \ldots, y_m .\, \phi_{\text{hard}}[X] \wedge \phi_{\text{easy}}[z, Y]$$

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- Formulas are *independent*.
- The solver never assigns Y variables.
- Several ways how to deal with this issue.

- Consider a formula $\exists \mathcal{X}\, \forall z \exists \mathcal{L} \,.\, \phi[\mathcal{X}, z, \mathcal{L}]$ such that proofs are exponential if they start on $\mathcal{L}$ first.

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- Prove that a DPLL-based solver resolves on $\mathcal{L}$ first.

THE FORMULA: COMPLETION PRINCIPLE

a_1	$\cdots$	a_1	$\cdots$	a_N	$\cdots$	a_N
b_1	$\cdots$	b_N	$\cdots$	b_1	$\cdots$	b_N

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- $\mathcal{X} = x_{11} \ldots x_{nn}$

a_1	$\cdots$	a_1	$\cdots$	a_N	$\cdots$	a_N
b_1	$\cdots$	b_N	$\cdots$	b_1	$\cdots$	b_N

- $\mathcal{X} = x_{11} \dots x_{nn}$
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a_1	$\cdots$	a_1	$\cdots$	a_N	$\cdots$	a_N
b_1	$\cdots$	b_N	$\cdots$	b_1	$\cdots$	b_N

- $\mathcal{X} = x_{11} \dots x_{nn}$
- $\mathcal{L} = a_1 \dots a_n, b_1 \dots b_n$
- $\exists \mathcal{X} \forall z \exists \mathcal{L}$

$$x_{ij} \vee z \vee a_i,\ i, j \in 1..n$$

$$\neg x_{ij} \vee \neg z \vee b_j,\ i, j \in 1..n$$

$$\bigvee_{i \in 1..n} \neg a_i$$

$$\bigvee_{i \in 1..n} \neg b_i$$

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COMPLETION PRINCIPLE: HARDNESS

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- Before a unit clause is learned, there is no *propagation across levels*, i.e. $\mathcal{L}$ variables are given a value only after all $\mathcal{X}$ variables are given a value.
- Derivations of the first unit clause is always by resolving $\mathcal{L}$ first.

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INTRICACIES

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n	CDCL	CDCL + SDCL	CDCL + SDCL − pure lits.
4	101	101	101
5	1081	1081	751
6	19611	19611	3531
7	370811	370811	36411
8	$>$ 9995451	$>$ 10000981	5464551
9	$>$ 10612011	$>$ 10619361	$>$ 931211
10	$>$ 10303551	$>$ 10313901	$>$ 8608251

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- Is the formula really hard for solution + conflict learning?
- Other methods in solving to break the order?

Thank You for Your Attention!

Questions?

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