On Checking of Skolem-based Models of QBF

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What Is QBF and Why Should We Certify Anything?

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- applications—model checking, fault localization, PSPACE-complete

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- QBF solvers are hard to write and easy to make mistakes in—certifying answers increases confidence
- certificates (proofs) can be useful for further analysis

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- Q-Resolution
- Term Resolution
- Skolem-based models (strategies)

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$$\forall y \exists x. \ (y \leftrightarrow x)$$

- the existential player always wins by playing x the same as y

Skolem-based Models

Strategy Function

for variable x, a Boolean function $f_x(y_1, \ldots, y_k)$, where $y_1, \ldots, y_k$ precede x in the prefix

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$$\forall y \exists x. \; (y \leftrightarrow x)$$

- $\{f_x(y) := y\}$

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\text{iff} & \bigwedge_{C \in \varphi} (\Omega_{\mathcal{M}} \rightarrow C) \qquad (\text{distribution of } \rightarrow) \\
\text{iff} & \text{for all } C \in \varphi, \mathrm{UNSAT}(\Omega_{\mathcal{M}} \wedge \neg C) \qquad (\xi \text{ iff } \mathrm{UNSAT}(\neg \xi))
\end{array}
$$

Basic algorithm

input : Φ, $\mathcal{M}$
output:

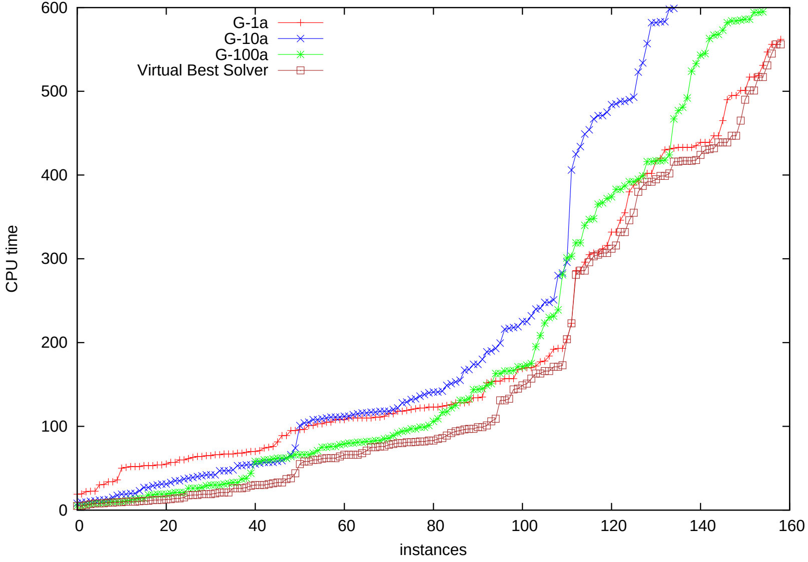
$\Omega_{\mathcal{M}} \leftarrow \texttt{CNF}(\mathcal{M})$ `// CNF representation of the functions`
forall the $C \in \varphi$ **do**
  if $\texttt{SAT}(\Omega_{\mathcal{M}} \wedge \neg C)$ **then** `// check the current clause`
    return false
  $\Omega_{\mathcal{M}} \leftarrow \Omega_{\mathcal{M}} \cup \{C\}$ `// implicate adding optimization`
return true

Clause-group algorithm

input : $\Phi = Q_1 z_1 \ldots Q_n z_n.\varphi$, $\mathcal{M}$ set of Skolem functions, $K \in \mathbb{N}^+$
output: true if $\mathcal{M}$ is model for Φ, false otherwise

```
ξ ← φ                                                  // clauses to check
Ω_M ← CNF(M)                     // CNF representation of the functions
while ξ ≠ ∅ do
    j ← min(|ξ|, K)
    γ ← pick j clauses from ξ                           // pick the group
    if SAT(Ω_M ∧ ⋁_{C∈γ} ¬C) then                   // check current group
        return false
    Ω_M ← Ω_M ∪ γ                  // implicates adding optimization
    ξ ← ξ ∖ γ
return true
```

Results



G-1a
G-10a
G-100a
Virtual Best Solver
CPU time
instances
0
100
200
300
400
500
600
0
20
40
60
80
100
120
140
160

Conclusions and Future Work

- algorithms for checking of QBF models (strategies)

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- instead of investigating one closet a time, investigate a group of clauses
- helps only in some cases
- in the future investigate heuristics for groups