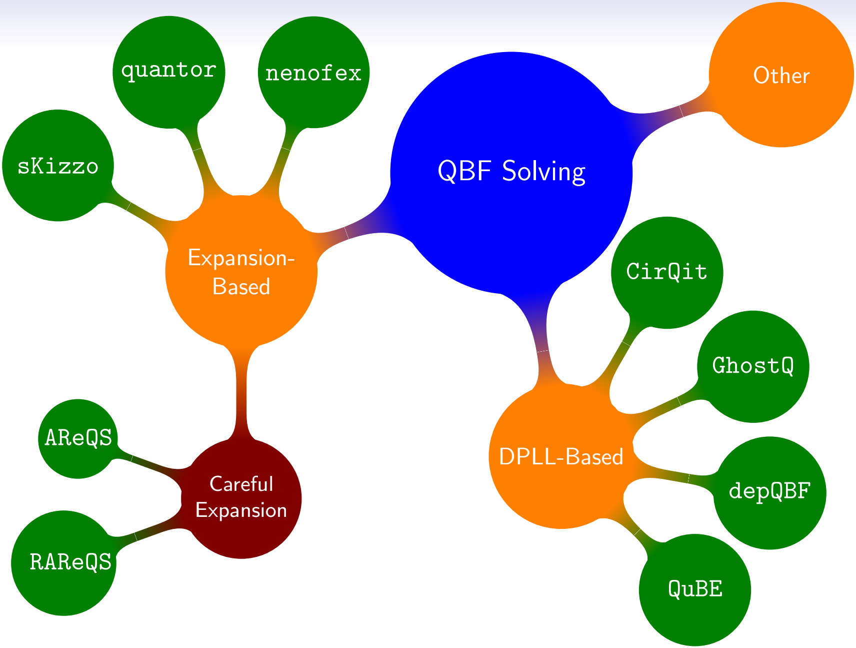
On Instantiation-Based Calculi for QBF

Mikoláš Janota[1] Olaf Beyersdorff[2] Leroy Chew[2]

[1] INESC-ID/IST, Lisbon, Portugal
School of Computing, University of Leeds, United Kingdom

QBF Workshop 2014, July 13


QBF Solving
Other
Expansion-Based
quantor
nenofex
sKizzo
Careful Expansion
AReQS
RAReQS
DPLL-Based
CirQit
GhostQ
depQBF
QuBE

Quantified Boolean Formula (QBF)

- an extension of SAT with quantifiers
  Example $\forall y_1 y_2 \exists x_1 x_2.\ (\bar{y}_1 \vee x_1) \wedge (y_2 \vee \bar{x}_2)$

Quantified Boolean Formula (QBF)

- an extension of SAT with quantifiers
  Example $\forall y_1 y_2 \exists x_1 x_2 .\ (\bar{y}_1 \vee x_1) \wedge (y_2 \vee \bar{x}_2)$

- we consider prenex form with maximal blocks of variables, and CNF matrix

$$\forall \mathcal{U}_1 \exists \mathcal{E}_2 \ldots \forall \mathcal{U}_{2N-1} \exists \mathcal{E}_{2N} .\ \phi$$

Quantified Boolean Formula (QBF)

- an extension of SAT with quantifiers
  Example $\forall y_1 y_2 \exists x_1 x_2.\ (\bar{y}_1 \vee x_1) \wedge (y_2 \vee \bar{x}_2)$
- we consider prenex form with maximal blocks of variables, and CNF matrix

$$\forall \mathcal{U}_1 \exists \mathcal{E}_2 \ldots \forall \mathcal{U}_{2N-1} \exists \mathcal{E}_{2N}.\ \phi$$

Solving and Proof Systems

- DPLL — Q-Resolution (`QuBE`, `depqbf`, etc.)
- Expansion — ∀Exp+Res (`Quantor`, `sKizzo`, `Nenofex`)
  - "Careful" expansion (`AReQS`,`RAReQS`)

Q-resolution

Q-resolution = *Q-resolution rule* + $\forall$*-reduction*

Q-resolution

Q-resolution = *Q-resolution rule* + $\forall$*-reduction*

Q-resolution rule

C_1, C_2 with one and only one complementary literal l, where l is existential

- derive $C_1 \cup C_2 \smallsetminus \{l, \bar{l}\}$

Q-resolution

Q-resolution = *Q-resolution rule* + $\forall$*-reduction*

Q-resolution rule

C_1, C_2 with one and only one complementary literal l, where l is existential

- derive $C_1 \cup C_2 \setminus \{l, \bar{l}\}$

$\forall$-reduction

- if $k \in C$ is universal with highest level in C, remove k from C

Q-resolution

Q-resolution $=$ *Q-resolution rule* $+$ $\forall$*-reduction*

Q-resolution rule

C_1, C_2 with one and only one complementary literal l, where l is existential

- derive $C_1 \cup C_2 \setminus \{l, \bar{l}\}$

$\forall$-reduction

- if $k \in C$ is universal with highest level in C, remove k from C

Tautologous resolvents are generally unsound and **not allowed**!

Expansion

$$\forall x.\ \Phi = \Phi[0/x] \wedge \Phi[1/x]$$

$$\exists x.\ \Phi = \Phi[0/x] \vee \Phi[1/x]$$

Expansion

$$\forall x.\ \Phi = \Phi[0/x] \wedge \Phi[1/x]$$

$$\exists x.\ \Phi = \Phi[0/x] \vee \Phi[1/x]$$

Fresh variables in order to keep prenex form

$$\exists e_1 \forall u_2 \exists e_3.\ (\bar{e}_1 \vee e_3) \wedge (\bar{e}_3 \vee e_1) \wedge (u_2 \vee e_3) \wedge (\bar{u}_2 \vee \bar{e}_3)$$

Expansion

$$\forall x.\ \Phi = \Phi[0/x] \wedge \Phi[1/x]$$

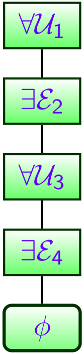
$$\exists x.\ \Phi = \Phi[0/x] \vee \Phi[1/x]$$

Fresh variables in order to keep prenex form

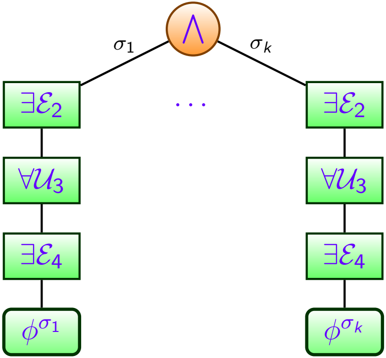
$$\exists e_1 \forall u_2 \exists e_3.\ (\bar{e}_1 \vee e_3) \wedge (\bar{e}_3 \vee e_1) \wedge (u_2 \vee e_3) \wedge (\bar{u}_2 \vee \bar{e}_3)$$

$$\begin{aligned} \exists e_1 e_3^{\bar{u}_2} e_3^{u_2}.\quad & (\bar{e}_1 \vee e_3^{\bar{u}_2}) \wedge (\bar{e}_3^{\bar{u}_2} \vee e_1) \wedge \\ & (\bar{e}_1 \vee e_3^{u_2}) \wedge (\bar{e}_3^{u_2} \vee e_1) \wedge \\ & e_3^{\bar{u}_2} \wedge \\ & \bar{e}_3^{u_2} \end{aligned}$$

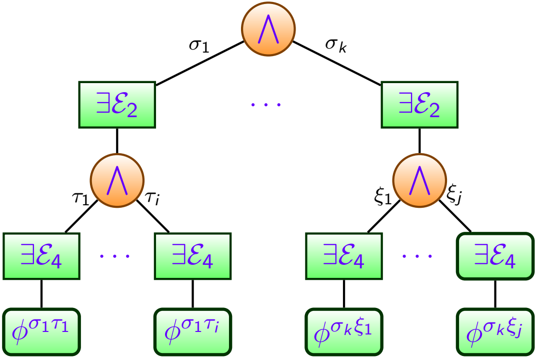
Recursive Partial Expansion



Recursive Partial Expansion



Recursive Partial Expansion



Different View on ∀Exp+Res

$$\frac{C \text{ in matrix}}{\left\{ l^{[\tau]} \mid l \in C, l \text{ is existential} \right\}} \text{(Axiom)}$$

- τ is a complete assignment to universal variables s.t. there is *no* $l \in C$ with $\tau(l) = 1$.
- $[\mu]$ takes only the part of μ that is $< l$

Different View on ∀Exp+Res

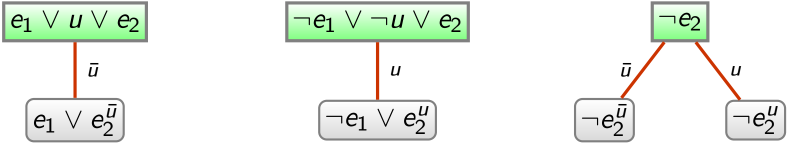
$$\frac{C \text{ in matrix}}{\left\{ l^{[\tau]} \mid l \in C, l \text{ is existential} \right\}} \text{ (Axiom)}$$

- τ is a complete assignment to universal variables s.t. there is *no* $l \in C$ with $\tau(l) = 1$.
- $[\mu]$ takes only the part of μ that is $< l$

$$\frac{x^{\tau} \vee C_1 \qquad \neg x^{\tau} \vee C_2}{C_1 \cup C_2} \text{ (Resolution)}$$

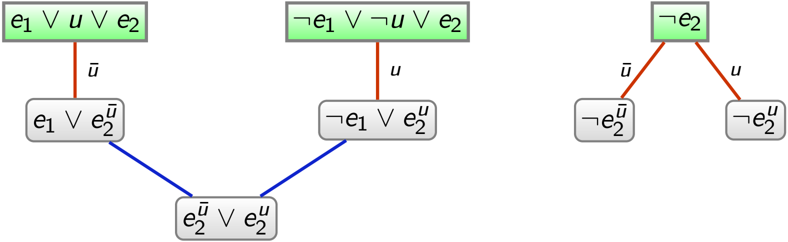
Example Proof in ∀Exp+Res

$\exists \mathbf{e_1} \forall \mathbf{u} \exists \mathbf{e_2}$



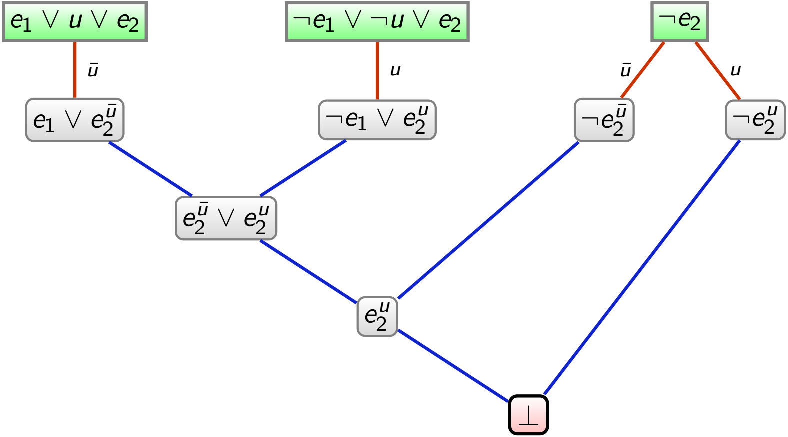
Example Proof in ∀Exp+Res

$\exists \mathbf{e_1} \forall \mathbf{u} \exists \mathbf{e_2}$



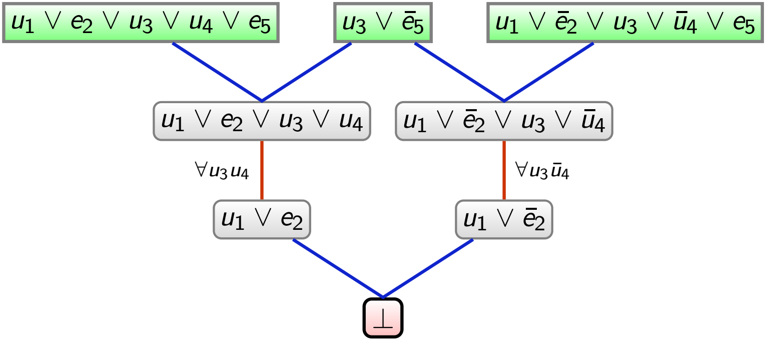
Example Proof in ∀Exp+Res

$\exists \mathbf{e_1} \forall \mathbf{u} \exists \mathbf{e_2}$



What is Hard for ∀Exp+Res

$\forall \mathbf{u_1} \exists \mathbf{e_2} \forall \mathbf{u_3 u_4} \exists \mathbf{e_5}$



IR-calc, "lazy instantiation"

$$\frac{C \text{ in matrix}}{\left\{x^{[\tau]} \mid x \in C, x \text{ is existential}\right\}}\ (\text{Axiom})$$

τ is the minimal assignment that assigns to 0 all universal literals of C

IR-calc, “lazy instantiation”

$$\frac{C \text{ in matrix}}{\left\{x^{[\tau]} \mid x \in C, x \text{ is existential}\right\}}\ \text{(Axiom)}$$

τ is the minimal assignment that assigns to 0 all universal literals of C

$$\frac{x^{\tau} \vee C_1 \qquad \neg x^{\tau} \vee C_2}{C_1 \cup C_2}\ \text{(Resolution)}$$

IR-calc, "lazy instantiation"

$$\frac{C \text{ in matrix}}{\left\{x^{[\tau]} \mid x \in C, x \text{ is existential}\right\}} \text{ (Axiom)}$$

τ is the minimal assignment that assigns to 0 all universal literals of C

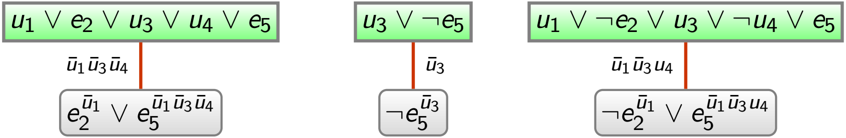
$$\frac{x^{\tau} \vee C_1 \qquad \neg x^{\tau} \vee C_2}{C_1 \cup C_2} \text{ (Resolution)}$$

$$\frac{l_1^{\tau_1} \vee \cdots \vee l_k^{\tau_k}}{l_1^{[\tau_1 \veebar \sigma]} \vee \ldots \vee l_k^{[\tau_k \veebar \sigma]}} \text{ (Instantiation by } \sigma\text{)}$$

$\tau_i \veebar \sigma$ "completes" τ_i with σ. E.g. $(\bar{u}_1 u_2 \veebar u_1 u_3) = \bar{u}_1 u_2 u_3$

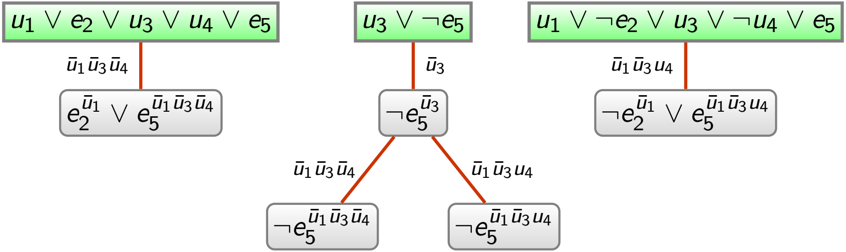
Example Proof in IR-calc

$\forall \mathbf{u_1} \exists \mathbf{e_2} \forall \mathbf{u_3} \forall \mathbf{u_4} \exists \mathbf{e_5}$



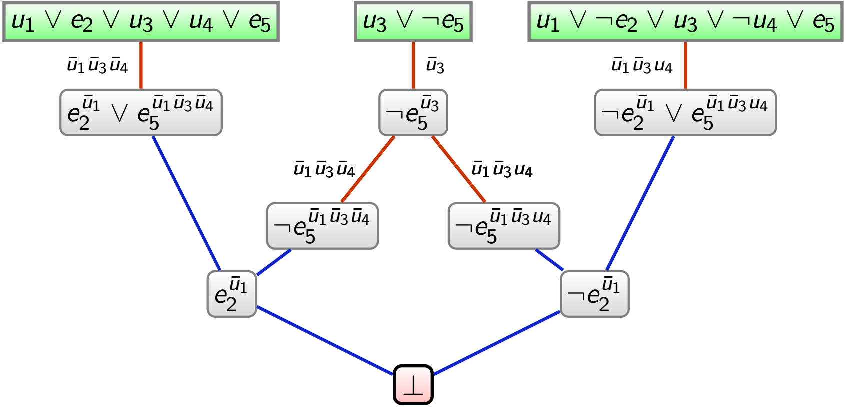
Example Proof in IR-calc

$\forall \mathbf{u_1} \exists \mathbf{e_2} \forall \mathbf{u_3} \forall \mathbf{u_4} \exists \mathbf{e_5}$



Example Proof in IR-calc

$\forall \mathbf{u_1} \exists \mathbf{e_2} \forall \mathbf{u_3} \forall \mathbf{u_4} \exists \mathbf{e_5}$



Simulation by IR-calc

- Since Q-Res does not allow tautologous clauses, a resolution $x^{\tau_1} \vee C_1$ and $x^{\tau_2} \vee C_2$ never has contradiction in τ_1, τ_2, which lets us make the annotation equal by instantiation.

Simulation by IR-calc

- Since Q-Res does not allow tautologous clauses, a resolution $x^{\tau_1} \vee C_1$ and $x^{\tau_2} \vee C_2$ never has contradiction in τ_1, τ_2, which lets us make the annotation equal by instantiation.
- To simulate ∀Exp+Res , immediately instantiate by the complete assignments used in the ∀Exp+Res proof.

Simulation by IR-calc

- Since Q-Res does not allow tautologous clauses, a resolution $x^{\tau_1} \vee C_1$ and $x^{\tau_2} \vee C_2$ never has contradiction in τ_1, τ_2, which lets us make the annotation equal by instantiation.
- To simulate ∀Exp+Res , immediately instantiate by the complete assignments used in the ∀Exp+Res proof.
- To simulate ∀-expansion

$$\forall u \exists x_1 \dots x_k . \phi \quad \rightsquigarrow \quad \exists x_1^{\bar{u}} x_1^{u} \dots x_k^{\bar{u}} x_k^{u} . (\phi^{\bar{u}} \wedge \phi^{u})$$

introduce two copies of ϕ by instantiating all clauses by u and $\bar{u}$, respectively (or do so lazily).

Summary and Future Work

- We have introduced an instantiation-based calculus IR-calc which simulates both Q-res and ∀Exp+Res.

Summary and Future Work

- We have introduced an instantiation-based calculus IR-calc which simulates both Q-res and ∀Exp+Res.
- Expansion can be done by considering only one polarity (instantiate by $\bar{u}$ only) but also by partial assignments.

Summary and Future Work

- We have introduced an instantiation-based calculus IR-calc which simulates both Q-res and ∀Exp+Res.
- Expansion can be done by considering only one polarity (instantiate by $\bar{u}$ only) but also by partial assignments.
- IR-calc simulates both Q-resolution and ∀Exp+Res but how does it compare to long-distance-Q-resolution?

Summary and Future Work

- We have introduced an instantiation-based calculus IR-calc which simulates both Q-res and ∀Exp+Res.
- Expansion can be done by considering only one polarity (instantiate by $\bar{u}$ only) but also by partial assignments.
- IR-calc simulates both Q-resolution and ∀Exp+Res but how does it compare to long-distance-Q-resolution?
- IR-calc simulates Q-res but is it more powerful?

Summary and Future Work

- We have introduced an instantiation-based calculus IR-calc which simulates both Q-res and ∀Exp+Res.
- Expansion can be done by considering only one polarity (instantiate by $\bar{u}$ only) but also by partial assignments.
- IR-calc simulates both Q-resolution and ∀Exp+Res but how does it compare to long-distance-Q-resolution?
- IR-calc simulates Q-res but is it more powerful?
- Strategy extraction exists [Beyersdorff et al., 2014]

Summary and Future Work

- We have introduced an instantiation-based calculus IR-calc which simulates both Q-res and ∀Exp+Res.
- Expansion can be done by considering only one polarity (instantiate by $\bar{u}$ only) but also by partial assignments.
- IR-calc simulates both Q-resolution and ∀Exp+Res but how does it compare to long-distance-Q-resolution?
- IR-calc simulates Q-res but is it more powerful?
- Strategy extraction exists [Beyersdorff et al., 2014]
- An extension of IR-calc exists that simulates long-distance-Q-resolution (IRM-calc) [Beyersdorff et al., 2014].

Thank you for your attention!

Questions?

Beyersdorff, O., Chew, L., and Janota, M. (2014).
On unification of QBF resolution-based calculi.
In *Mathematical Foundations of Computer Science (MFCS)*.
to appear.