On Unification of QBF Resolution-Based Calculi

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  (4) 1 (True)

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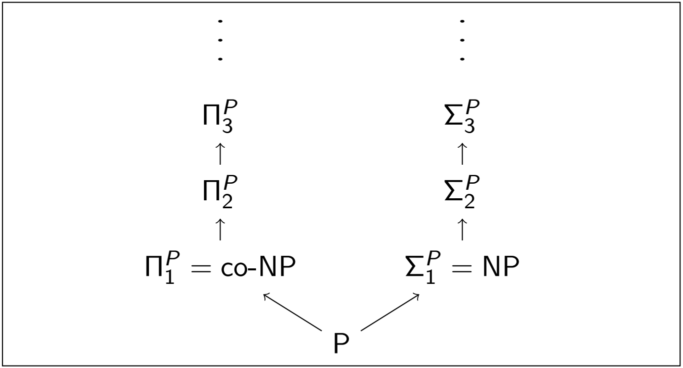

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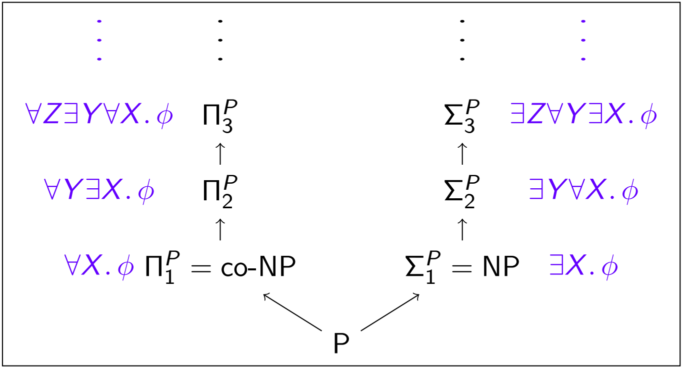

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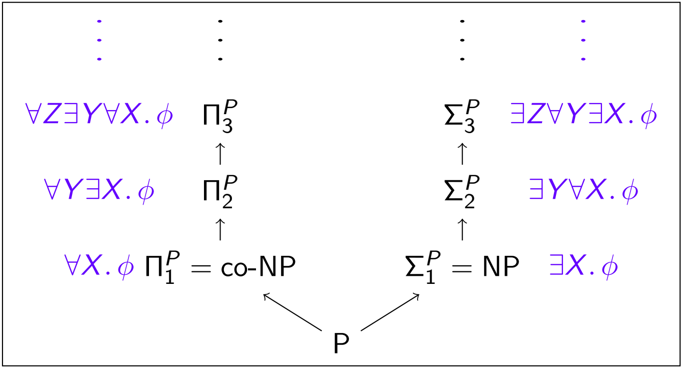

- Deciding QBF is PSPACE complete

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$$\forall u \exists e.\, (u \vee e) \wedge (\bar{u} \vee \bar{e})$$

$\exists$ wins by playing $e \leftarrow \bar{u}$.

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Proof systems and solving

- DPLL (`QuBE`, `depqbf`, etc.)
- Expansion (`AReQS`, `RAReQS`, `Quantor`, `sKizzo`, `Nenofex`)
- unification, certification, understanding of QBF solvers

Q-resolution [Büning et al., 1995]

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long-distance Q-resolution [Balabanov and Jiang, 2012] enables tautologous resolvents in *some cases*.

Q-resolution Example

$$\forall \mathbf{u} \exists \mathbf{e}.\ (\mathbf{u} \vee \bar{\mathbf{e}}) \wedge (\mathbf{u} \vee \mathbf{e})$$

$u \vee \bar{e}$ $\qquad$ $u \vee e$

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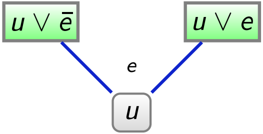

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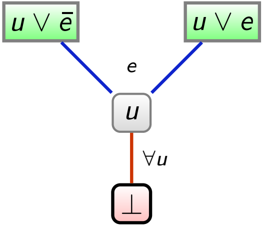

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Fresh variables in order to keep prenex form

$$\exists e_1 \forall u_2 \exists e_3.\ (\bar{e}_1 \vee e_3) \wedge (\bar{e}_3 \vee e_1) \wedge (u_2 \vee e_3) \wedge (\bar{u}_2 \vee \bar{e}_3)$$

Expansion

$$\forall x.\ \Phi = \Phi[0/x] \land \Phi[1/x]$$

Fresh variables in order to keep prenex form

$$\exists e_1 \forall u_2 \exists e_3.\ (\bar{e}_1 \lor e_3) \land (\bar{e}_3 \lor e_1) \land (u_2 \lor e_3) \land (\bar{u}_2 \lor \bar{e}_3)$$

$$\begin{aligned}\exists e_1 e_3^{\bar{u}_2} e_3^{u_2}.\ \ & (\bar{e}_1 \lor e_3^{\bar{u}_2}) \land (\bar{e}_3^{\bar{u}_2} \lor e_1) \land \\ & (\bar{e}_1 \lor e_3^{u_2}) \land (\bar{e}_3^{u_2} \lor e_1) \land \\ & e_3^{\bar{u}_2} \land \\ & \bar{e}_3^{u_2}\end{aligned}$$

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  Refute as propositional (propositional resolution). Effectively this means we can use a SAT solver.

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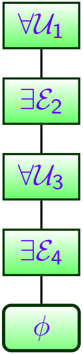

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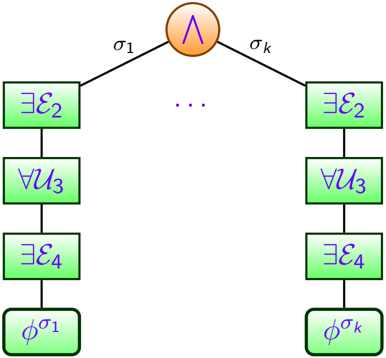

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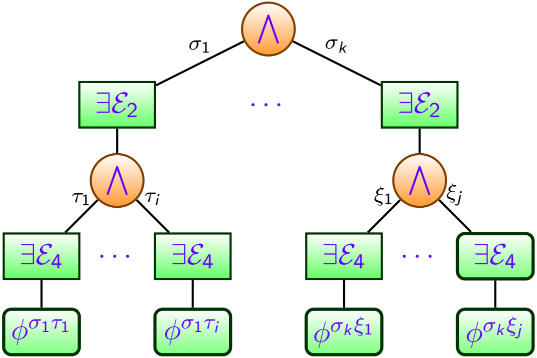

∀Exp+Res [J. and Marques-Silva, 2013]

$$\frac{C \text{ in matrix}}{\left\{ l^{[\tau]} \mid l \in C, l \text{ is existential} \right\}} \text{ (Axiom)}$$

- τ is a complete assignment to universal variables s.t. there is *no* $l \in C$ with $\tau(l) = 1$.
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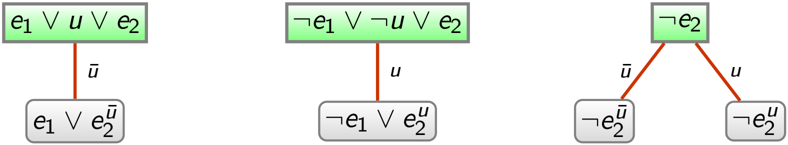
Example Proof in $\forall$Exp+Res

$\exists \mathbf{e_1} \forall \mathbf{u} \exists \mathbf{e_2}$



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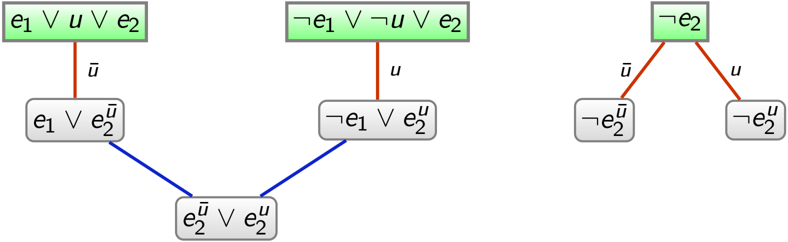

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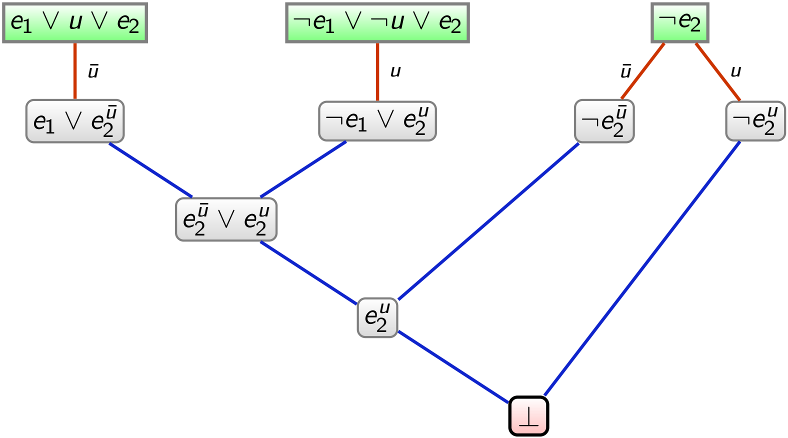
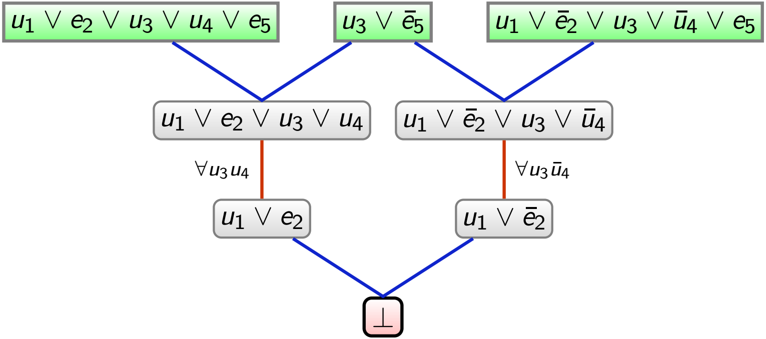


What is Hard for ∀Exp+Res

$\forall \mathbf{u_1} \exists \mathbf{e_2} \forall \mathbf{u_3 u_4} \exists \mathbf{e_5}$



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- IR-calc p-simulates Q-Resolution and ∀Exp+Res.
- IRM-calc additionaly p-simulates Long-distance Q-Resolution.
- For both IR-calc and IRM-calc we show polynomial winning-strategy extraction from IR-calc and IRM-calc refutations.

IR-calc

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$$\frac{l_1^{\tau_1} \vee \cdots \vee l_k^{\tau_k}}{l_1^{[\tau_1 \veebar \sigma]} \vee \ldots \vee l_k^{[\tau_k \veebar \sigma]}} \text{ (Instantiation by } \sigma\text{)}$$

$\tau_i \veebar \sigma$ "completes" τ_i with σ. E.g. $(\bar{u}_1 u_2 \veebar u_1 u_3) = \bar{u}_1 u_2 u_3$

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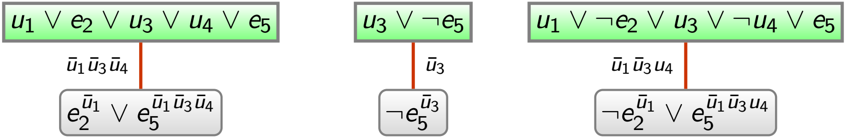

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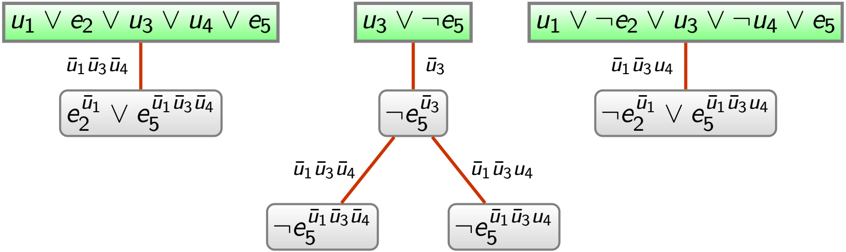

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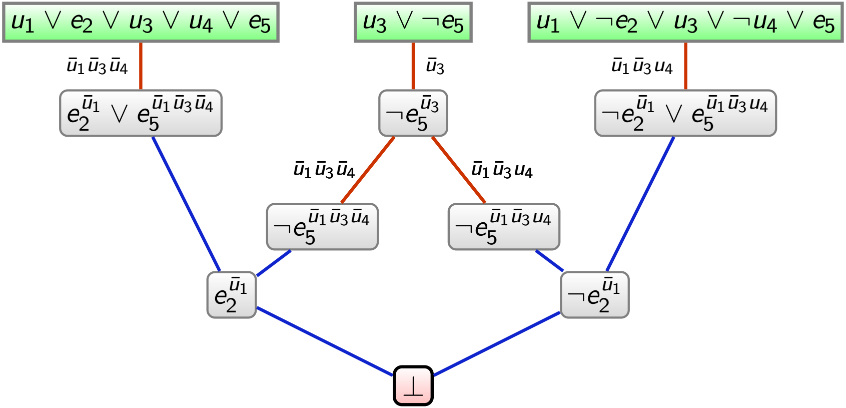

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- To simulate ∀Exp+Res, immediately instantiate by the complete assignments used in the ∀Exp+Res proof.

IRM-calc— “Merging” calculus

- enables *merging* $b^{1/u} \vee b^{0/u}$ into $b^{*/u}$.
- However, restricts resolution, $b^{*/u}$ and $\neg b^{c/u}$ cannot be resolved.

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$$\frac{x^{\tau \cup \xi} \vee C_1 \qquad \neg x^{\tau \cup \sigma} \vee C_2}{\mathrm{inst}(\sigma, C_1) \cup \mathrm{inst}(\xi, C_2)} \text{ (Resolution)}$$

$\mathrm{dom}(\tau)$, $\mathrm{dom}(\xi)$ and $\mathrm{dom}(\sigma)$ are mutually disjoint. $\mathrm{rng}(\tau) = \{0, 1\}$

$$\frac{C \vee b^{\mu} \vee b^{\sigma}}{C \vee b^{\xi}} \text{ (Merging)}$$

$\mathrm{dom}(\mu) = \mathrm{dom}(\sigma)$
$\xi = \{c/u \mid c/u \in \mu, c/u \in \sigma\} \cup \{*/u \mid c/u \in \mu, d/u \in \sigma, c \neq d\}$

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- IR-calc simulates Q-res but is it more powerful?

Summary and Future Work

- Proof systems for QBF are not yet well understood.
- We have introduced novel systems based on expansion: IR-calc, IRM-calc.
- IR-calc simulates both Q-resolution and ∀Exp+Res.
- IRM-calc additionally simulates long-distance Q-resolution.
- A ∀ winning strategy can by extracted in polynomial time.
- Expansion can be done by considering only one polarity (instantiate by $\bar{u}$ only) but also by partial assignments.
- IR-calc simulates Q-res but is it more powerful?
- Is IRM-calc needed?

Thank you for your attention!

Questions?

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