On QBF Proofs and Preprocessing

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Quantified Boolean Formula (QBF)

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- we consider prenex form with CNF matrix

$$\forall \mathcal{U}_1 \exists \mathcal{E}_2 \ldots \forall \mathcal{U}_{2N-1} \exists \mathcal{E}_{2N}. \phi$$

- prefix: $\forall \mathcal{U}_1 \exists \mathcal{E}_2 \ldots \forall \mathcal{U}_{2N-1} \exists \mathcal{E}_{2N}$
- matrix: ϕ

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Research Question

How to provide proofs in the context of preprocessing?

Approach

Instance

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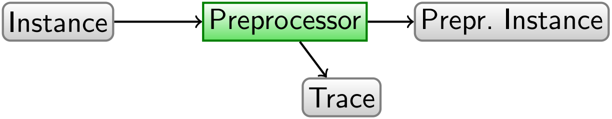

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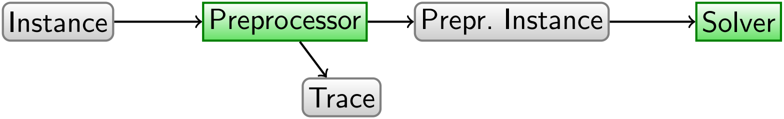

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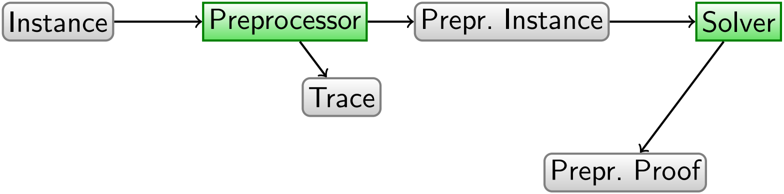

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Prepr. Instance
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Trace
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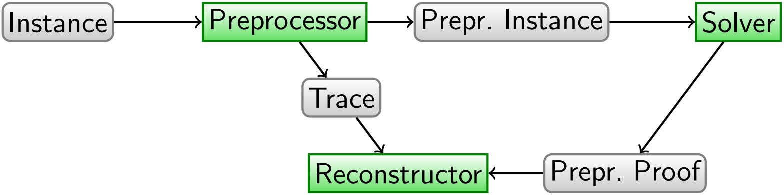

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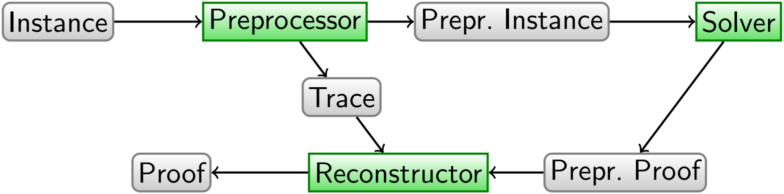

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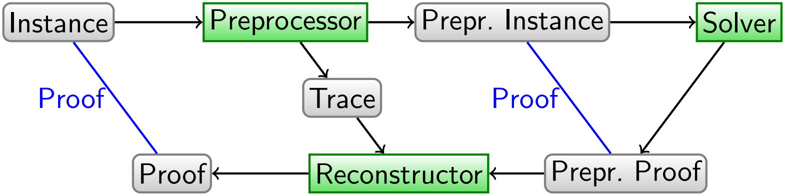
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DPLL-based QBF Solving

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Expansion-based QBF Solving

$\forall$Exp+Res—seems incomparable to Q-resolution [Janota and Marques-Silva, 2013]

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* A side-condition is needed for soundness.

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- For true QBF we focus on Models (strategies) instead.
- Q-resolution is sufficient to reconstructed considered techniques.

A Few Words about Reconstructions

- Reconstruction done "backwards". For preprocessings $P_1, \ldots, P_n$ and respective reconstructions $R_1, \ldots, R_n$ we do:

  $P_n(\ldots(P_2(P_1(\Psi)))\ldots)$, where Ψ is a formula
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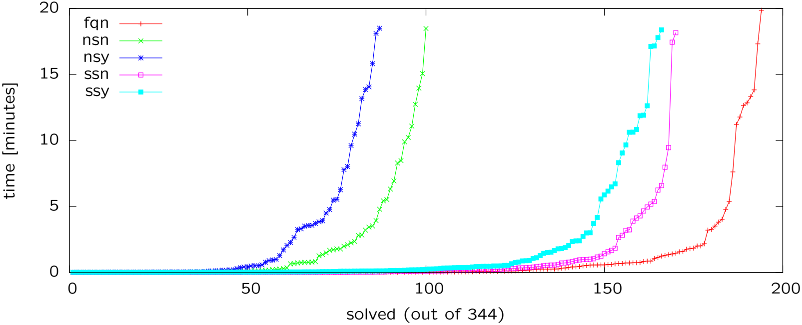
  - In both cases, all literals are blocked in the "classical sense".

Experimental Evaluation



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- For future: More preprocessing techniques.
- How to polynomially certify preprocessing for true QBFs?

Thank you for your attention!

Questions?

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