SAT Solving in Interactive Configuration

by

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A Thesis submitted to the
University College Dublin
for the degree of Doctor of Philosophy

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November 2010

Contents

Abstract

During configuration a user is looking for a desired solution to a set of constraints. This process of finding a solution becomes *interactive* if the user is obtains assistance throughout the process from a dedicated tool. Such tool is called a *configurator*.

Most of the up-to-date research advocates implementing configurators using *precompilation*. The motivation for precompiling the configured instance is to obtain a guaranteed response time during configuration processes. However, it is well-known that precompilation approaches scale poorly. Scalability is an inherent problem in precompilation, since precompilation approaches explicitly represent all of the possible solutions of the configured instance. Hence, precompiling larger instances requires the use of heuristics and expertise.

Motivated by these problems with precompilation, this dissertation focuses on a different approach: a *lazy approach*, where computation is carried out only when needed.

The domain of investigation is interactive configuration of *propositional instances*, i.e., where variables take either the value TRUE or FALSE. The precompilation technology used in this context is predominantly binary decision diagrams (BDDs). The underlying technology this dissertation builds on is *satisfiability (SAT) solvers*, which are tools that decide whether a given propositional formula has a solution or not.

The use of SAT solvers is motivated by the significant advances of SAT solving that took place in the last two decades. In particular, modern SAT solvers offer the following advantages over BDDs:

- Many solvers are available freely for download.
- Modern solvers have proven to scale very well; benchmarks in recent SAT competitions have up to 10 million variables.
- SAT solving is a highly active research field with new extensions appearing each year. These extensions improve the computation time as well as enable solving more types of problems. Such extensions represent potential for configuration of non-propositional logics.

The goal of this dissertation is to show that SAT-based interactive configuration does not suffer from scalability problems and it provides response times that do not inconvenience the user. Moreover, the dissertation shows that the SAT-based approach enables providing more informative explanations than the precompiled approaches.

The main contributions of this dissertation are the following:

- algorithms to implement an interactive configurator using a SAT solver;
- algorithms to produce explanations for actions performed by the configurator;
- an optimization that reduces the size of the given formula in the context of interactive configuration and how to *reconstruct explanations* in the context of such optimizations;
- a formalization of the completion of a configuration process and what it means to *make a choice for the user*;
- an algorithm that helps the user to *complete* a configuration process *without making a choice* for the user;
- a description of the design and implementation of the presented techniques;
- an empirical evaluation of the implementation.

Typesetting Environments Several typesetting environments are used throughout the text. The following conventions are used for them.

- *Definitions* are formal definitions of concepts used later in the text.
- *Observations* are facts that are very easy to prove or are obvious without a proof.
- *Lemmas* are facts that are used to prove some other facts.
- *Propositions* are facts that may be important but are typically easy to prove.
- *Claims* are substantial facts.
- *Remarks* are notes without which the surrounding text still makes sense. They are aimed at giving to the reader more intuition of the discussed material.
- *Dings*, marked with ⬇, are short, typically informal, conclusions of the preceding text.

Chapter 1

Introduction

Computer users encounter configuration on daily basis. Whether when they customize an application they use, customize how a new application is installed, or just customize a query to a search engine. In all these cases the user picks values from a set of choices and receives feedback when these choices are invalid. And, in all these cases the underlying application or query is being configured.

Configuration can be done interactively or non-interactively. In the interactive case, the computer provides information about validity of choices each time the user makes a new choice—this feedback typically takes the form of graying out the choices that are no longer valid. In the non-interactive case, the user first makes all the choices first and the computer checks the validity and highlights the violations at the very end.

While non-interactive configuration is easier to implement for programmers, interactive configuration is more user-friendly. It can be easily argued that fixing a form after some fields have been marked as red (invalid) is far more tedious than obtaining the feedback instantaneously.

A configurator is a tool that enables interactive configuration and it is interactive configuration that is the main focus of this dissertation. In particular, the focus is on the reasoning that a configurator has to perform in order to infer which choices are valid and which are not.

The objective of this chapter is to familiarize the reader with the context of configuration and with the basic principles used in configurators. In particular, it discusses reuse (Section 1.1), modularization (Section 1.2, and software product line engineering (Section 1.3). Section 1.4 presents a configurator from a user perspective. To conclude the introduction, Section 1.5 discusses the motivation and lists the main contributions of the dissertation. Section 1.7 outlines the organization of the text.

1.1 Reuse

Software development is hard for many reasons: humans err, forget, have trouble communicating, and software systems are larger than what one human brain can comprehend. Researchers in Software Engineering try to help software developers by looking for means that improve quality and efficiency of software development. *Reuse* is one of these means and it is the main motivation for configuration.

The basic idea of reuse is simple: developers use a certain product of their work for multiple purposes rather than providing a new solution in each case. This concept is not limited to source code. *Artifacts* that can be reused include models, documentation, or build scripts. These artifacts are examples of a more general class: programs written in a domain specific language (*DSL*) to produce or generate other entities.

Let us briefly look at the main benefits of successful reuse. Firstly, it takes less time to develop an artifact once than multiple times. Secondly, fixing, evolving or modifying an artifact is more efficient than fixing multiple occurrences of it. Last but not least, code more used is code more tested, and, code more tested is code more reliable.

There are several obstacles to reuse. A reusable piece of software needs to provide more functionality than when it is developed for a single purpose. Hence the developer must come up with a suitable *abstraction* of the software—if the developer has not envisioned the future use of the software right, other developers will not be able to reuse it.

Another obstacle is that if a developer writes software from scratch, it can be expected that this developer understands it quite well. On the other hand, reusing code developed by other people requires an investment into learning about the artifacts to be reused. And, clearly: "*For a software reuse technique to be effective, it must be easier to reuse the artifacts than it is to develop the software from scratch*" [121]. Alas, to determine whether it is easier to develop from scratch or to reuse is hard.

Frakes and Kang highlight several other issues with nowadays reuse [72]. One issue is the scalability of reuse. Indeed, it is not always easy for a programmer to find the right component for the activity in question if the component repository is large. To address this problem a number of techniques have been proposed. Such as *semantic-based component retrieval* [202], *component rank* [99], or by providing information delivery systems that help programmers with learning about new features [203]. Another problem highlighted by Frakes and Kang is that state-of-the-practice means for describing components is insufficient. They propose *design by contract* as an answer to this issue [149, 123]

1.2 Modularization

One important step toward reuse is *modularization* [56]. In a nutshell, modularization is dividing artifacts (and the problem) into clearly defined parts while inner workings of these parts are concealed as much as possible. For this purpose, most mainstream programming languages provide means for *encapsulation* and encapsulated components communicate via *interfaces*.

Modularization follows the paradigm of *separation of concerns*: it is easier to focus on one thing than on many at the same time (human brain has limited capacity). At the same time, modularization facilitates reuse by making it easier to determine the portion of an existing artifact that needs to be reused in order to obtain the desired functionality.

Software engineers, when writing modularized code, deal with an inherent problem: how to describe an artifact well without relying on its inner workings? In other words, devising component interfaces is difficult. Mainstream high-level languages (C-family, ML family) provide syntax-based interfaces embellished with type-checking. Such interfaces typically do not carry enough information for the component to be usable. Hence, natural language documentation is an imperative.

Natural language documentation, however, is not machine-readable and thus it cannot be used for reasoning about the component nor it is possible to automatically check whether the implementation fulfills the documentation. Logic-based annotations try to address this issue. Indeed, languages like Java Modeling Language (JML) [123] are a realization of Floyd-Hoare triples [96]. The research in this domain is still active and to this date the approach has not achieved a wide acceptance.

To conclude the discussion on modularization we should note that the concept of a module differs from language to language. Many researchers argue that the traditional approaches to modularization are weak when it comes to orthogonal separation of concerns. *Feature Oriented Programming* (FOP) [168, 17, 16, 129] and Aspect oriented programming (AOP) [117] are examples of approaches to orthogonal modularization at the source-code level.

1.3 Software Product Line Engineering

Software Product Line Engineering (SPLE) [39] is a Software Engineering approach which tries to improve reuse by making it systematic. More specifically, a software product line (or just product line) is a system for developing a particular set of software products while this set is explicitly defined. A popular means

for defining such set are feature models, which are discussed in Section 2.4. This set needs to comprise products that are similar enough for the approach to be of value. For this purpose we use the term *program families*, defined by Parnas as follows.

> ... sets of programs whose common properties are so extensive that it is advantageous to study the common properties of the programs before analyzing individual members. [165]

With a bit of imagination, program families appear even in single-product development as eventually any successful single program ends up having several versions. In SPLE, however, the family of products to be targeted is defined up front and is *explicitly* captured in *variability models*. The name variability models comes from the fact that such model must capture how the products in the family differ from one another. Analogously, properties shared among the products form the *commonality*[1]. The analysis concerned with determining the variability model is called *domain analysis*.

The motivation for deciding on the product family beforehand is so that the individual products can be manufactured with minimal effort. An individual product of the family is developed with the reuse of *core assets* — a set of artifacts that is to be reused between members of the family. In the ideal case, products are assembled only using the core assets with little additional work needed. The construction toy *Lego* is a popular metaphor for this ideal case: The individual Lego pieces are core assets reused to obtain new constructs.

Variability models and the set of pertaining potential products are often generalized to the concepts of *problem space* and *solution space* [44]. Problem space is formed by descriptions of the products that might or should be developed and solution space by potential implementations of products. In terms of these concepts, a task of a product line is to map elements of problem space to elements of solution space (see Figure 1.1). Figure 1.1 shows the role of a

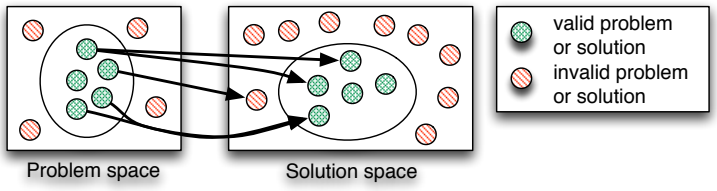


Figure 1.1: Concepts in a product line

variability model in a product line: the rectangles represent whole spaces, the

[1] Admittedly, these terms are typically defined vaguely.

ellipses represent parts of the determined by the model. The model determines a set of *valid products*, which is a subset of a larger space.

To relate to our Lego example, the pieces enable us to assemble all sorts of constructions but we are probably interested only in those that do not keel over. Analogously, typically only a subset of the problem space is considered. We might, for example, require that we are only interested in those compositions of core assets that compile.

- *A software product line connects a solution space and a problem space; the valid set of problems is explicitly defined by a variability model.*

We should note that often solution and problem space are in a 1-to-1 relationship and the distinction between them is not immediately clear. For instance, a core asset repository may contain a component `Optimizer` that provides the optimization functionality. The functionality is part of the problem space (Do we want an optimized product?) while the component belongs into the solution space. The distinction would have been more apparent if we had components `OptimizerA` and `OptimizerB` both providing the optimization functionality. A formalization of the relation between the two spaces is found in [104].

- *SPLE is a systematic approach to reuse which explicitly defines the set of potential products.*

1.4 A Configurator from the User Perspective

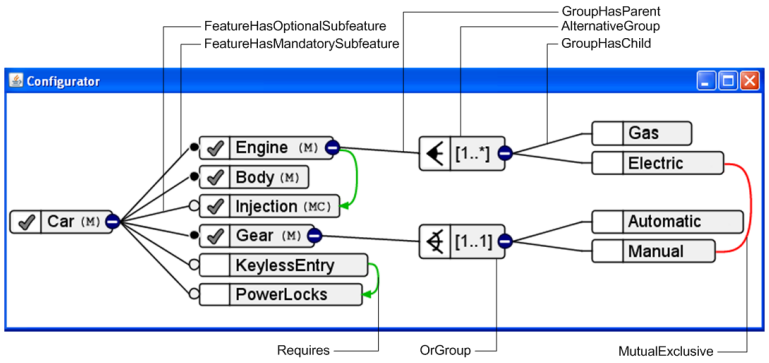


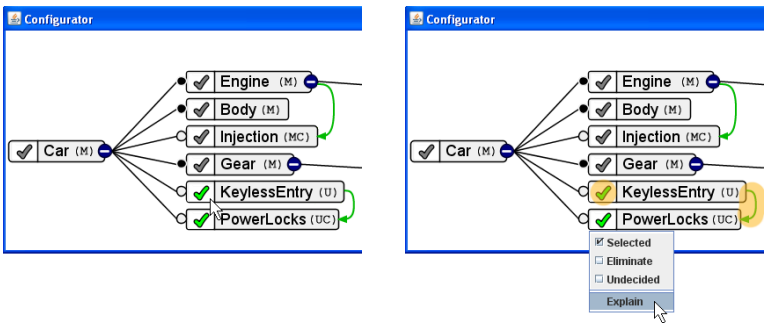
Figure 1.2: Visual representation of a feature model in Lero S^2T^2

This section discusses the main properties that a user is expecting from a configurator. Broadly speaking, a configurator is a tool that supports configuration of variability models. Even though these variability models typically have an underlying mathematical semantics, the models are represented in some domain specific modeling language.

Figure 1.2 shows a screenshot from the configurator Lero S^2T^2 [174], which is using a reasoning backend hinging on the algorithms described in this dissertation (see Chapter 7 for more details about the tool). The instance being configured in the screenshot is a car with eleven features where each feature is represented by a unique name in a rectangle. The features are organized in a tree hierarchy rooted in the feature `Car`. The features `Engine`, `Body`, and `Gear` are mandatory, which is depicted by a filled circle at the end of the edge connecting the feature to the parent feature. The features `Injection`, `KeylessEntry`, and `PowerLocks` are optional, which is depicted by an empty circle. The green edges capture the requirements that `Engine` *requires* `Injection` and that `KeylessEntry` *requires* `PowerLocks`. At least one of the features `Gas` and `Electric` must be selected because they are in an *or-group*, which is denoted by the cardinality `1..*`. Exactly one of the features `Automatic` and `Manual` must be selected because they are in an *alternative group*, which is denoted by the cardinality `1..1`. The red edge captures the requirement that the features `Manual` and `Electric` are mutually exclusive. A collection of features together with the dependencies between them is called a *feature model*.

This notation originates in the paper of Kang et al. entitled *"Feature-oriented domain analysis (FODA) feasibility study"* and thus the notation is known as the FODA feature modeling language [114]. The precise notation and semantics of the language is discussed in Section 2.4.

When configuring a feature model, a user assembles the product by choosing the required features of the final product while preserving the dependencies in the model. The configurator enables a user to *select* or *eliminate* a feature, meaning that the feature must or must not appear in the final configuration, respectively. We refer to selecting or eliminating a feature as *user decisions*. A user may decide to *retract* a user decision at a later stage.



(a) automatic selection of a feature

(b) explanation of an automatic selection.

Figure 1.3: Feedback from the configurator

A configurator prevents the user from making invalid choices, choices that would violate the feature model, by automatically selecting or eliminating features. For instance, in Figure 1.3a the user has selected the feature `KeylessEntry` and the configurator automatically selected the feature `PowerLocks` since the car cannot have `KeylessEntry` without `PowerLocks`.

In some cases, the user wants to know why a certain feature was automatically selected or eliminated. A piece of information showing why a certain automatic choice was made is called an *explanation*. An example of a situation where an explanation is useful is when the user wants to select a certain feature but that is not possible because that feature was automatically eliminated. Then, the user can change the choices that led to the automatic elimination of that feature. Analogously, the user might want to eliminate a certain feature but that is not possible because it has been automatically selected.

Figure 1.3b depicts an example of an explanation. In this example the user has asked why power-locks were automatically selected. The configurator highlights the *requires* edge between `KeylessEntry` and `PowerLocks`, and, the user decision to select `KeylessEntry`. We can imagine, for example, that the user does not want `PowerLocks` because they are too expensive. The explanation, however, shows to the user that `KeylessEntry` cannot be selected without `PowerLocks`. Then, the user might either decide to pay for the `PowerLocks` or sacrifice `KeylessEntry`, i.e., eliminate the feature `KeylessEntry`.

Note that by automatically eliminating or selecting certain features the configurator effectively disallows certain user decisions. If the feature had been automatically eliminated, the user cannot select the feature and vice versa. Configurators may differ in which user decisions they disallow. The focus of this dissertation is a configurator that is *backtrack-free* and *complete*.

Backtrack-freeness means that if the user is making only allowed decisions then it will always be possible to complete the configuration without retracting some decisions (without backtracking). Completeness of a configurator means that the configurator disallows only decisions that are necessary to disallow. In other words, any feature configuration permitted by the feature model is reachable by a sequence of allowed user decisions in a complete configurator. Both of these concepts, backtrack freeness and completeness, are defined formally in Section 3.1.

Configuration does not have to be limited to feature models. Nor do the user decisions have to be limited to selection and elimination. For instance, the configurator may support numerical variables and user decisions that affect their values. In this dissertation, however, the primary focus is a configuration supporting selections and eliminations. The interested reader is referred to the chapter concerned with the related work for further references on configuration

of other types of constraints (Chapter 10).

1.5 Motivation and Contributions

The majority of research on configuration hinges on precompilation. In precompilation the instance to be configured is precompiled into a specialized data structure, which enables implementing a configurator with guaranteed response time [3]. In *propositional* interactive configuration, which is the focus of this dissertation, the data structure predominantly used for precompilation is *binary decision diagrams* (*BDDs*) [92, 89, 5, 196, 93, 94].

In contrast to precompilation, the approach presented in this dissertation is a *lazy approach*—no precompilation is required as all the computation takes place as the configuration proceeds.

There are two main arguments for researching the lazy approach: *scalability* and *informativeness of explanations*. It is well-known that BDDs are difficult to construct for instances with more than hundreds of variables [1, 23]. The reason for poor scalability of BDDs is the fact that a BDD is an *explicit* representation of all the satisfying assignments of the pertaining formula. Since the number of satisfying assignments of a formula grows exponentially with the number of variables, it is easy to encounter formulas whose BDD representation is beyond the capacities of modern computers [33]. Moreover, finding a BDD that is *optimal* in size is hard [26]. Consequently, precompilation of large instances requires heuristics specific to the instance. Clearly, this is highly impractical for laymen users.

The problem of scalability of precompilation is inherent because precompilation explicitly represents all the solutions of the configured instance, and, the number of solutions typically grows exponentially with the number of variables. A number of researchers propose a variety of heuristics and modifications of the data structures for precompilation in order to deal with scalability [136, 164, 189, 187, 188, 148]. While some of these heuristics and approaches may succeed for a particular instance, some may not. This poses a significant technological barrier: the user has to experiment with the different approaches to see which work, and, these techniques are often intricate and therefore not easy to implement. We argue that the SAT approach *scales uniformly*, i.e., there is no need for heuristics for specific instances. Additionally, from an engineering perspective, the use of a SAT solver as the underlying technology is an advantage because many SAT solvers are freely available and new ones appear each year. The research in SAT solving is propelled by the yearly SAT competitions [176, 177]. In contrast to that, BDD libraries are scarce and the last known comparative study of BDD libraries is from the year 1998 [201].

The second argument for our approach is that precompilation methods have limited capabilities when it comes to explanations. Again, this problem is inherent because the structure of the instance that is to be configured is lost in the precompiled data structure, i.e., the mapping from formulas to BDDs is *not* injective. Even though there exists work on providing proofs using BDDs, it is not clear how explanations would be constructed from these proofs (see Chapter 9 for further discussion). Consequently, the BDD-based approaches provide only explanations as sets of user decisions. In contrast to that, in our approach we provide not only the user decisions but also the parts of the configured instance and dependencies between them. We argue that this adds significant value to the explanations because in order to understand why a certain choice was disabled, the user needs to take into account the configured instance.

Hence, the main contribution of this dissertation is a demonstration of the advantages of the SAT-based approach to implementing interactive configuration over the precompilation-based approach. The particular contributions are the following:

- Chapter 4 describes algorithms for implementing a configurator using a SAT solver and Section 4.5 describes algorithms for constructing explanations from resolution trees obtained from the solver.
- Chapter 5 shows how to integrate into a configurator a syntactic-based optimization and how to reconstruct proofs in the presence of this optimization.
- Chapter 6 formalizes what it means to *"make a choice for the user"* and Section 6.3 presents an algorithm for computing variables that can be eliminated without making a choice for the user.
- Chapter 7 describes a design of the implemented configurator, which suggests design patterns that other programmers could find useful in implementing configurators.
- Chapter 8 empirically evaluates the presented algorithms and shows the impact of the different optimizations.

1.5.1 Comparison to State-of-the-Art

Interactive Configuration

Even though some aspects of the lazy approach are addressed in the existing research literature, a thorough study of the subject is missing. Kaiser and Küchlin propose an algorithm for computing which variables must be set to TRUE and

which to FALSE in a given formula [113]. This is, in fact, a sub-problem of propositional interactive configuration, as in interactive configuration, the formula effectively changes as the user makes new decisions. Additionally to the work of Kaiser and Küchlin, this dissertation shows how to take advantage of the computations made for the previous steps of the configuration process.

The basic ideas behind the presented algorithms is captured in an earlier article of mine [103]. The works of Batory and Freuder et al. provide algorithms for providing lazy approach to interactive configuration [79, 77, 15]. The work of Freuder et al. is in the context of constraint satisfaction problems (CSPs) [79, 77] while the work of Batory is in the context of propositional configuration [15]. Even though the approaches are lazy, as the one presented in this dissertation, they do not satisfy the requirements of *completeness* and *backtrack-freeness*. Informally, completeness is when the user can reach all possible solutions and backtrack-freeness is when the configurator allows only such sequences of user decisions that can be extended into a solution (these concepts are formally defined in Chapter 3 and the above-mentioned articles are discussed in greater detail in Chapter 10).

The approach of Batory is inspired by *truth maintenance systems* [52, 69], in particular by *logic truth maintenance systems* (*LTMSs*). An LTMS is a helper-component for algorithms that traverse a state space in order to find a solution; such traversal may be carried out by backtracking, for instance. One role of an LTMS is to identify if the search reached a conflict, i.e., identify that the traversing algorithm is in a part of the state space that does not contain a solution. The second role is to identify parts of the state space that should not be traversed because they do not contain a solution; this is done by unit propagation (see Section 2.1.3). In Batory's approach it is not an algorithm that traverses the state space, but it is the user who does so. However, the role of the LTMS remains the same.

An LTMS is implicitly contained in our approach because a SAT solver performs unit propagation in order to prune the state space[2] (see Section 2.3). On a general level, an interactive configurator satisfies the interface of an LTMS. However, guaranteeing backtrack-freeness is as computationally difficult as finding a solution (see Section 4.3.2). Hence, using a backtrack-free interactive configurator as a component for an algorithm that is looking for a solution in the search space is not practical.

Finally, we note that the lazy approach to configuration also appears in *configuration of non-discreet domains*. Indeed, it is unclear how to precom-

[2]Modern SAT solvers do not use an LTMS as a separate component but instead tie the implementation of the functionality of an LTMS with the search algorithm. This enables a more efficient implementation of the functionality because the order of the traversal is known.

pile configuration spaces with infinite numbers of solutions. Compared to our problem domain (propositional configuration), non-discreet domains are substantially different. Hence, the underlying technology has to be substantially different as well. In particular, the technology used in the reviewed literature are procedures from *linear programming* [131, 90].

Explanations

The algorithms constructing explanations from resolution trees (Section 4.5) are novel as the surveyed literature only shows how to provide explanations as sets of user decisions in backtrack-free and complete configurators. Interestingly enough, the explanations provided in non-backtrack-free approaches are also resolution trees [79, 15]. However, since these algorithms are profoundly different from those presented in this dissertation, their mechanism of explanations cannot be reused (see also Section 4.5.3 for further discussion).

Syntactic Optimization

The syntactic optimization (BIS-optimization) proposed in Chapter 5 is not novel. Indeed, it is well-known in the SAT community that the computation of satisfiability can be optimized using this technique (see Section 10.2 for further details). However, that are two main contributions that this dissertation makes in the context of this optimization.

The first contribution is our *novel manner of application*. Applying the BIS-optimization at the level of a SAT solver and at the level of a configurator is not the same. In particular, it does not yield the same effect—as the configurator calls the solver multiple times, the optimization has greater effect than if it were applied individually whenever the solver is called. Hence, the optimization is applied in a different fashion than in SAT solving. Moreover, the same effect achieved by the proposed technique cannot be achieved by adding the optimization to the underlying solver (see Remark 5.21 for further details).

The second contribution in regards to the BIS-optimization is *proof reconstruction*. The surveyed literature does not contain any work on how to reconstruct resolution trees obtained from a solver that operates on the optimized formula (which is needed to provide explanations). Additionally, we provide a heuristic that aims at minimizing the size of the reconstructed tree. Since the empirical evaluation shows that this heuristics significantly reduces the size, this serves as motivation for studying similar heuristics for other types of syntactic optimizations.

Configuration Process Completion

The formalization of the concept "making a choice for the user" appears in an earlier article of mine [105] and does not appear in any other surveyed literature. Compared to that article, the definition presented in this dissertation is improved for clarity and intuition (the concepts of choice were not explicit in the earlier definitions). More importantly, the dissertation shows how to calculate variables that can be eliminated without making a choice for the user.

Implementation

The described architecture of the implemented configurator (Chapter 7) is the well-known pipe-and-filter architecture and also bears some characteristics of the GenVoca approach [18], as the translation from a modeling language to a logic language is performed in a chain of smaller steps (in GenVoca these steps are called *layers*). However, there are some particularities in the design—mainly, in the way how explanations are dealt with. In order to enable explanations, the configured instance is not treated as a whole, but as a set of *primitives*, and the components realizing the translation from a modeling language to a logic language need to communicate in both directions.

- *The dissertation studies a lazy approach to interactive configuration—an approach in which computation is done only when needed. The underlying technology for realizing this approach are modern satisfiability (SAT) solvers.*

1.6 Thesis Statement

SAT solvers and BDDs are two predominant alternative means for solving propositional problems. BDDs were invented in 1986 by Bryant [34] and celebrated a great success in model checking in the early 90s [35, 144]. Efficient SAT solvers started to appear somewhat later, i.e., in the late 90s [138, 139, 204, 158]. Shortly after that, SAT solvers were successfully applied in various domains [181, 183, 25, 14, 128, 154, 59, 53, 28]. In particular, it has been shown that in model checking SAT solvers do not suffer from scalability issues as BDDs do [23, 1].

These success stories of SAT solvers served as the main motivation for this dissertation: investigate how SAT solvers can be applied in the context of interactive configuration. This direction of research is further encouraged by recent work by Mendonça et al. who show that SAT solving in the context of feature models is easy despite the general NP-completeness of the problem [147].

While interactive configuration has been mainly tackled by BDD-based approaches, this dissertation argues that SAT-based approach scales better and enables more informative explanations. Hence, the outlined arguments lead to the following thesis statement.

> *SAT solvers are better for implementing a Boolean interactive configurator than precompiled approaches.*

1.7 Outline

The dissertation is divided into the following chapters. Chapter 2 introduces notation and terms that are used in the dissertation. This chapter can be skipped and used as a back reference for the other chapters.

Chapter 3 puts configuration in a formal context and derives several important mathematical properties related to configuration. These properties are used by Chapter 4 to construct a configurator using a SAT solver. Chapter 5 proposes an optimization that is based on processing the instance to be configured; several important issues regarding explanations are tackled.

Chapter 6 studies how to help the user to *complete* a configuration process without making a choice for the user. An implementation of the proposed technique using a SAT solver is described.

Chapter 7 describes the design of the configurator that was implemented using the presented algorithms. The chapter discusses how a feature model is represented and the architecture is organized.

Chapter 8 presents measurements that were performed on several sample instances. The response time of the computation is reported and discussed. Additionally, for the technique that is designed to help the user to complete a configuration it is reported how efficient it is.

Chapter 10 overviews the related work and Chapter 11 concludes the dissertation.

Chapter 2

Background

This chapter serves as a quick reference for background material of this dissertation. Section 2.1 overviews general computer science terms. In particular Section 2.1.6 lists the notation used throughout the dissertation. Section 2.2 references the technology that can be used to implement the algorithms presented in the dissertation. Section 2.3 describes the main principles behind modern SAT solvers. Section 2.4 defines the notation and semantics for feature models that are used as instances for configuration.

2.1 General Computer Science

2.1.1 Propositional Logic

Most of the text in this thesis operates with *propositional logic* with standard Boolean connectives ($\wedge, \vee, \Rightarrow, \Leftrightarrow$) and negation ($\neg$) with their standard meaning. The names of propositional variables are typically lower-case letters taken from the end of the alphabet.

For a set of Boolean variables, a *variable assignment* is a function from the variables to $\{\text{TRUE}, \text{FALSE}\}$. A variable assignment is often represented as the set of variables that are assigned the value TRUE, the other variables are implicitly FALSE. A *model* of a formula ϕ is such a variable assignment under which ϕ evaluates to TRUE (a model can be seen as a solution to the formula).

Example 1. *The formula $x \vee y$ has the models $\{x\}$, $\{y\}$, and $\{x, y\}$. The assignment $\emptyset$, which assigns* FALSE *to all variables, is not a model of $x \vee y$.*

An implication $x \Rightarrow y$ corresponds to the disjunction $\neg x \vee y$. For the implication $x \Rightarrow y$, the implication $\neg y \Rightarrow \neg x$ is called its *contrapositive*. An implication and its contrapositive are equivalent (the sets of models are the same).

A formula is *satisfiable* iff there is a assignment of the variables for which the formula evaluates to TRUE (the formula has a model); a formula is *unsatisfiable* iff it is not satisfiable. If a formula evaluates to TRUE under all variable assignments then it is called a *tautology*. For instance, $x \vee y$ is satisfiable, $x \wedge \neg x$ is unsatisfiable, and $x \vee \neg x$ is a tautology. We say that the formulas ψ and ϕ are *equisatisfiable* iff they are either both satisfiable or both unsatisfiable.

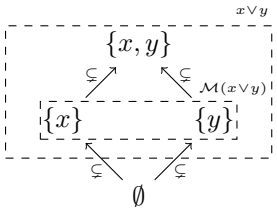
We write $\psi \models \phi$ iff ϕ evaluates to TRUE under all variable assignments satisfying ψ, e.g., $x \wedge y \models x$. We write $\psi \not\models \phi$ if there is a variable assignment under which ψ is TRUE and ϕ is FALSE. Note that for any tautology ϕ, it holds that TRUE $\models \phi$ and for any formula ψ it holds that FALSE $\models \psi$.

2.1.2 Minimal Models

A model of a formula ϕ is *minimal* iff changing a value from TRUE to FALSE for any subset of the variables yields a non-model. When models are represented as sets of values whose value is TRUE, then changing a value from TRUE to FALSE corresponds to removing the variable from the model. Therefore a model M of ϕ is minimal iff any proper subset of M is not a model of ϕ.

We write $\mathcal{M}(\phi)$ to denote the set of all minimal models of ϕ. We write $\phi \models_{\min} \psi$ to denote that ψ holds in all minimal models of ϕ.

Example 2. *The formula $\phi \stackrel{\text{def}}{=} x \vee y$ has the models $\{x\}, \{y\}, \{x, y\}$. The model $\{x, y\}$ is not a minimal model of $x \vee y$ because removing either x or y yields a model. However, removing x from $\{x\}$ yields $\emptyset$, which is not a model of ϕ and therefore $\{x\}$ is a minimal model of ϕ. Analogously, $\{y\}$ is a minimal model. Therefore $\mathcal{M}(x \vee y) = \{\{x\}, \{y\}\}$. Since $\{x, y\}$ is not a minimal model, it holds that $x \vee y \models_{\min} \neg(x \wedge y)$.*



The different sets of variables form a lattice ordered by the subset relation. Models of a particular formula are a subset of that lattice (the outer rectangle). The minimal models are the smallest elements in that subset (the inner rectangle).

Figure 2.1: Ordering on models

Figure 2.1 gives another insight into the concept of minimal models. The subset relation defines an ordering where the empty set is the smallest element and the set of all the considered variables the largest. Minimal models are the minimal elements of this ordering on the models of the formula in question.

Minimal models are used in Chapter 6 in order to find variables that can

be set to FALSE without making a choice for the user (see Definition 16). In particular, we will look for variables that have the value FALSE in all minimal models (see Section 6.3).

2.1.3 Conjunctive Normal Form (CNF)

A propositional formula is called a *literal* iff it is a variable or a negated variable (e.g., $\neg v$). A formula is called a *clause* iff it is a disjunction of zero of more literals (e.g., $\neg v \vee w$, $\neg x$). Each variable must appear at most once in a clause. A formula is in the ***c**onjunctive **n**ormal **f**orm (CNF)* iff it is a conjunction of clauses, e.g., $(\neg v \vee w) \wedge (\neg x)$. For a literal l, we write $\bar{l}$ to denote the *complementary literal*, i.e., if $l \equiv x$ then $\bar{l} \stackrel{\text{def}}{=} \neg x$; if $l \equiv \neg x$ then $\bar{l} \stackrel{\text{def}}{=} x$.

The notion of literal and clause is of particular importance for understanding the dissertation.

When a formula is in CNF, it is often treated as a set of clauses (implicitly conjoined). Analogously, a clause can be seen as literals (implicitly disjoined). However, clauses are typically written as disjunctions.

Special forms of clauses The *empty clause*, i.e., clause containing no literals, is denoted as $\bot$ and is logically equivalent to FALSE. A clause comprising one literal is called a *unit clause*. Any implication $l \Rightarrow k$ where l and k are literals, corresponds to the clause $\bar{l} \vee k$. Therefore, we write such implications in a CNF formula whenever suitable.

A clause c_1 is *subsumed* by the clause c_2 iff the set of literals of c_2 is a subset of the literals of c_1 (e.g., x subsumes $x \vee y$). If a set of clauses contains such two clauses, removing c_1 yields an equisatisfiable formula, e.g., $(x \vee y) \wedge x$ is equisatisfiable with x. (Moreover, any satisfying assignment of x can be extended to a satisfying assignment of $(x \vee y) \wedge x$ by assigning an arbitrary value to the variable y.)

Resolution

Two clauses c_1 and c_2 can be *resolved* if they contain at least one mutually complementary literal. To *resolve* such c_1 and c_2 means to join the clauses while omitting the literal and its complement. More precisely, for clauses comprised of the sets of literals c_1 and c_2, respectively, such that $l \in c_1$ and $\bar{l} \in c_2$ for some variable v that $\{v, \neg v\} = \{l, \bar{l}\}$, *resolution over the variable* v yields a clause comprising the literals $c_1 \cup c_2 \setminus \{v, \neg v\}$. The result of resolution is called the *resolvent*. For instance, resolving $x \vee y$ and $x \vee \neg y$ yields the resolvent x.

A resolution of two clauses for which at least one is a unit clause is called a

unit resolution. Performing unit resolution exhaustively is called *unit propagation*.

Example 3. *Unit propagation on the clauses* $\{p, \neg p \vee q, \neg q \vee x\}$ *yields the set* $\{p, \neg p \vee q, \neg q \vee x, q, x\}$. *Since* $\neg p \vee q$ *is subsumed by* q *and* $\neg q \vee x$ *is subsumed by* x, *the following set of clauses is equisatisfiable* $\{p, q, x\}$.

Proving by Resolution

In logic in general, proofs are carried out from a theory where a theory is a set of formulas. Since any CNF formula is a set of clauses, i.e., smaller formulas, the difference between a theory and a formula is only conceptual. Hence, in this dissertation CNF formulas are treated as theories whenever needed.

We say that ψ is *proven* from ϕ *by resolution* if $\bot$ is derived from $\phi \wedge \neg\psi$ by a finite number of applications of the resolution rule; we write $\phi \vdash \psi$ if that is the case.

The *resolution* rule is a *sound* inference rule, i.e., if r is a resolvent of c_1 and c_2, it holds that $c_1 \wedge c_2 \models r$. *Resolution* is *complete* with respect to consistency, i.e., ψ is unsatisfiable (inconsistent), if and only if $\bot$ can be derived from ψ by a finite number of applications of the resolution rule. Consequently, the resolution proof system is sound and complete, i.e., $\phi \vdash \psi$ if and only if $\phi \models \psi$.

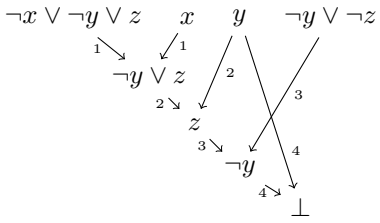
A set of resolutions deriving $\bot$ from a formula ϕ form a tree with some of the clauses from ϕ as leaves and $\bot$ as the root; we refer to such tree as *resolution tree*. We say that such tree *refutes* ϕ.

Example 4. *Let* ϕ *be* $((x \wedge y) \Rightarrow z) \wedge x \wedge (\neg y \vee \neg z)$, *then* $\phi \vdash \neg y$. *To show that* $\phi \vdash \neg y$, *we refute* $\phi \wedge y$. *This refutation is realized by the resolution tree in Figure 2.2. Note that resolution trees are typically drawn upside down compared to trees in general.*

> ***Remark 2.2.*** *The graph in Figure 2.2 is rather a directed acyclic graph (DAG) than a tree because some clauses are used in multiple resolution steps. Nevertheless, the graph can be redrawn as a tree if all these clauses are duplicated each time they are used. We mostly use the DAG representation since it is more compact but we continue using the term tree.*

> ***Remark 2.3.*** *Resolution can generalized to formulas in first order logic but this is not used in this dissertation. For details on resolution see [132].*

Example 5. *Figure 2.3 presents three common patterns of resolution. (1) Resolving* x *with* $x \Rightarrow z$ *yields* z *because* $x \Rightarrow z$ *is equivalent to* $\neg x \vee z$. *In general,*



An example of a resolution tree where the edges are labeled with the number of the pertaining resolution step, e.g., the resolution of y and $\neg y$ which yields $\bot$ is the last (fourth) step.

Figure 2.2: An example resolution proof

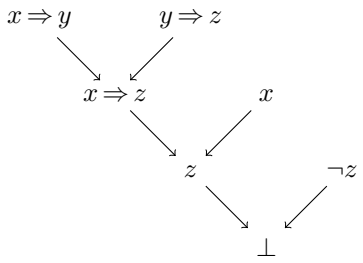


Figure 2.3: Example resolution with implications

resolving a unit clause with an implication resembles modus ponens. (2) Resolving $x \Rightarrow y$ with $y \Rightarrow z$ yields $x \Rightarrow z$. In general, resolving two-literal clauses resembles the transitivity law on implications. (3) The only way how a resolution step can yield the empty clause ($\bot$) is that both of the resolved clauses are unit clauses and they comprise mutually complementary literals (such as z and $\neg z$).

In some cases we operate on a tree of resolutions where the root is not $\bot$ but some clause c; in this more general case we use the term *tree of resolutions*. Note that for any tree of resolutions, the root clause is a consequence of the formula comprising the leaves of the tree. More generally, any node in the tree as a consequence of the set of leaves reachable from that node. As an example, Figure 2.4 shows a tree of resolutions deriving w from the set of clauses $\{\neg z \vee \neg y \vee w, z, \neg z \vee y\}$. Hence, it holds $\neg z \vee \neg y \vee w, z, \neg z \vee y \models w$. Also, $\neg z \vee \neg y \vee w, z \models \neg y \vee w$ and $\neg z \vee y, z \models y$.

Conceptually, showing that $\phi \models \psi$ by refuting $\phi \wedge \neg\psi$ corresponds to *proof by contradiction*. Using resolution to derive some clause as in Figure 2.4 corre-

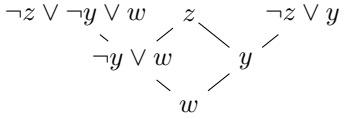


Figure 2.4: Tree of resolutions deriving w

sponds to traditional *proof by deduction*. The reason why proof by contradiction is used with resolution is that resolution is not complete for deduction but only for contradiction. For instances, there is no way how to show $x \models x \vee y$ by applying the resolution rule to the clause x. However, it is possible to refute $x \wedge \neg x \wedge \neg y$.

CNF and complexity

To determine whether a CNF formula is satisfiable or not is NP-complete [41]. However, even though deciding whether a formula is a tautology is in general *co-NP-complete*, a CNF formula is a tautology if and only if it comprises only clauses that contain a literal and its complement, which can be detected in polynomial time.

Converting to CNF

Any formula can be equivalently rewritten into CNF using *De Morgan laws*. This may, however, cause an exponential blow-up in the size of the formula. For instance, applying De Morgan's laws to the formula $(x_1 \wedge y_1) \vee \cdots \vee (x_n \wedge y_n)$ results into 2^n clauses each having n literals.

Nevertheless, it is possible to transform any formula into a CNF formula which is equisatisfiable and linear in the size of the original one by using *Tseitin transformation* [192].

Tseitin transformation adds new variables to represent subformulas and adds clauses to record the correspondence between these variables and the subformulas. Here we show Tseitin transformation for a language with $\neg$, $\wedge$, and $\vee$. The reader is referred for example to a monograph by Bradley and Manna for further details [31].

Let $R(\phi)$ be a variable representing the subformula ϕ where each variable in the original formula is represented by itself ($R(v) = v$) while all the other representative variables are fresh. For the formula $\phi \equiv \psi \wedge \xi$ add the clauses $R(\phi) \Rightarrow R(\psi)$, $R(\phi) \Rightarrow R(\xi)$, and $(R(\psi) \wedge R(\xi)) \Rightarrow R(\phi)$. For the formula $\phi \equiv \psi \vee \xi$ add the clauses $R(\phi) \Rightarrow (R(\psi) \wedge R(\xi))$, $R(\psi) \Rightarrow R(\phi)$, and $R(\xi) \Rightarrow R(\phi)$. For the formula $\phi \equiv \neg\psi$ add the clauses $R(\phi) \Rightarrow \neg R(\psi)$ and $\neg R(\psi) \Rightarrow R(\phi)$. Finally, to assert that the formula must hold, add the clause $R(\phi)$ where ϕ is the whole formula being encoded.

Example 6. *Consider the formula* $(x_1 \wedge y_1) \vee \cdots \vee (x_n \wedge y_n)$. *For each term* $x_i \wedge y_i$ *introduce an auxiliary variable* r_i *and generate the clauses* $\neg r_i \vee x_i \vee y_i, \neg x_i \vee r_i, \neg y_i \vee r_i$. *Together with these clauses an equisatisfiable representation of the original formula in CNF is* $r_1 \vee \cdots \vee r_n$.

Although Tseitin transformation preserves satisfiability, some properties it does **not** preserve. One particular property not preserved by the transformation is the set of minimal models as shown by the following example.

Example 7. *Let $\phi \equiv \neg x \vee z$ and let n be the variable representing $\neg x$ and f be the variable representing ϕ. The Tseitin transformation results in the following set of clauses.*

$$\begin{array}{cc} f \Rightarrow (n \vee z) & x \vee n \\ n \Rightarrow f & \neg x \vee \neg n \\ z \Rightarrow f & f \end{array}$$

While ϕ has the only minimal model $\emptyset$, i.e., assigning False *to both x and z, the transformed formula has the minimal models $\{f, n\}$ and additionally the minimal model $\{f, z, x\}$.*

2.1.4 Constraint Satisfaction Problem (CSP)

Most of the text of this dissertation deals with propositional logic. This section discusses several basic concepts from constraint satisfaction programming, which are referenced mainly in the related work section.

Definition 1 (CSP). *A* CSP *is a triple $\langle \mathcal{V}, \mathcal{D}, \mathcal{C} \rangle$ where $\mathcal{V}$ is a set of variables $\mathcal{V} = \{v_1, \ldots, v_n\}$, $\mathcal{D}$ is a set of their respective finite domains $\mathcal{D} = \{D_1, \ldots, D_n\}$. The set $\mathcal{C}$ is a finite set of constraints. A constraint $C \in \mathcal{C}$ is a pair $\langle V_C, D_C \rangle$ where $V_C \subseteq \mathcal{V}$ is the* domain *of the constraint and D_C is a relation on the variables in V_C, i.e., $D_C \subseteq D_{i_1} \times \cdots \times D_{i_k}$ if $V_C = \{v_{i_1}, \ldots, v_{i_k}\}$.*

A variable assignment *is an n-tuple $\langle c_1, \ldots, c_n \rangle$ from the Cartesian product $D_1 \times \cdots \times D_n$, where the constant c_i determines the value of the variable v_i for $i \in 1 \ldots n$.*

A variable assignment is a solution *of a CSP if it satisfies all the constraints.*

For an overview of methods for solving CSP see a survey by Kumar [122].

Arc Consistency A CSP is *arc consistent* [74] iff for any pair of variables and any values from their respective domains these values satisfy all the constraints on those variables. The term *arc* comes from the fact that we are ensuring that for any pair of values there is an "arc" between them that records the fact that this pair of values satisfies the pertaining constraints.

Arc consistency can be achieved by removing values from domains of variables as illustrated by the following example.

Example 8. *Let us have a CSP with variables x_1 and x_2 and their respective domains $D_1 \stackrel{\text{def}}{=} \{1\}$ and $D_2 \stackrel{\text{def}}{=} \{0, 1\}$. And let us consider the constraint $\{\langle 1, 1 \rangle\}$ which enforces both variables to be* 1.

The CSP is not arc consistent because the value 0 *for* x_2 *never satisfies the constraint. However, removing* 0 *from its domain yields an arc consistent CSP.*

The algorithm AC-3 reduces domains of variables in order to achieve arc consistency in time polynomial to the size of the problem [135].

> ***Remark 2.4.*** *Arc consistency is in fact a special case of k-consistency [74], which takes into account k variables; arc consistency is k-consistency for* $k = 2$*. If a problem is k-consistent for k equal to the number of variables of the CSP, then for each domain and a value there exists a solution of the CSP using that value. Any k less than that does not guarantee this property. Nevertheless, ensuring k-consistency removes values that are certainly not used in any solution [74].*

2.1.5 Relation between CSP and Propositional Logic

Any propositional formula can be expressed as a CSP by considering all the variable domains to be $\{\textsc{False}, \textsc{True}\}$.

Example 9. *Let us have the variables* x*,* y*,* z*, and the formula* $(x \Rightarrow y) \wedge (y \Rightarrow z)$*. The formula can be expressed as the two following constraints each corresponding to one of the implications.*

$$c_1 \stackrel{\text{def}}{=} \langle \{x, y\}, \{\langle \textsc{False}, \textsc{False}\rangle, \langle \textsc{False}, \textsc{True}\rangle, \langle \textsc{True}, \textsc{True}\rangle\}\rangle$$
$$c_2 \stackrel{\text{def}}{=} \langle \{y, z\}, \{\langle \textsc{False}, \textsc{False}\rangle, \langle \textsc{False}, \textsc{True}\rangle, \langle \textsc{True}, \textsc{True}\rangle\}\rangle$$

If arc consistency is combined with *node consistency*—a requirement that no value violates any unary constraint—then it is equivalent to unit propagation on 2-literal clauses (see Section 2.1.3) as illustrated by the following example.

Example 10. *Consider again the formula in the example above (Example 9) represented as a CSP but now with the additional unary constraint requiring* x *to be* True*.*

Since the value False *violates the unary constraint, it is removed from the domain of* x*. Consequently, the constraint on* x *and* y *will be pruned to contain only* $\langle$True, True$\rangle$ *and the value* False *will be removed from the domain of* y *(see below). Analogously,* False *will be removed from the domain of* z*.*

x	x	y	y	z
True	~~False~~	~~False~~	~~False~~	~~False~~
	~~False~~	~~True~~	~~False~~	~~True~~
	True	True	True	True

2.1.6 Notation

$\langle e_1, \ldots, e_n \rangle$	n-tuple with the elements $e_1, \ldots, e_2$
$\{x \mid p(x)\}$	set comprehension: the set of elements satisfying the predicate p
$x \stackrel{\text{def}}{=} e$	x is defined to be e
$x \equiv e$	x is syntactically equivalent to e
$\phi \models \psi$	ϕ evaluates to TRUE in all models of ψ (Section 2.1.1)
$\phi \vdash \psi$	there exists a proof for ψ from ϕ (Section 2.1.3)
$\bar{l}$	the complementary literal of the literal l (Section 2.1.3)
$\mathcal{M}(\phi)$	the set of minimal models of ϕ (Section 2.1.2)

2.2 Technology

2.2.1 SAT Solvers

In the context of propositional logic, the *satisfiability problem* (*SAT*) is the problem of determining for a given formula whether there is an assignment such that the formula evaluates to TRUE or not. A *SAT solver* is a tool that decides the satisfiability problem. Most SAT solvers accept the input formulas in CNF and that is also assumed in this dissertation.

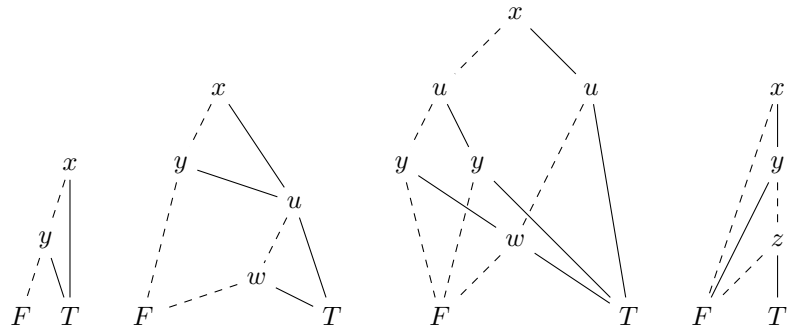
If a SAT solver is given a satisfiable formula, the solver returns a satisfying assignment (a model) of the formula. If the given formula is unsatisfiable, the solver returns a proof for the unsatisfiability.

The principles under which most modern SAT solvers are constructed enable them to produce resolution proofs of unsatisfiability (see Section 2.1.3). This is achieved by recording certain steps of the solver and then tracing them back [206].

Since a resolution proof may be very large in size, some solvers record the proof on disk in a binary format, e.g., MiniSAT [151]. In such case, to obtain the proof one needs to read it from disk and decode it.

Some solvers do not produce a resolution-based proof but return only a subset of the given clauses that is unsatisfiable. The solver SAT4J [178] is an example of a solver producing such set upon request (using the QuickXplain algorithm [111]). Since SAT4J is used in the implementation presented in this dissertation, another solver is used to obtain the resolution tree from the leaves obtained from SAT4J (see Section 7.1 for more details).

Despite the NP-completeness of the satisfiability problem, nowadays SAT solvers are extremely efficient and are able to deal with formulas with thousands of variables and are still improving as illustrated by the yearly *SAT Competition* [176].



(a) A representation of $x \vee y$

(b) A representation of $(x \vee y) \wedge (u \vee w)$ where the subtree for $u \vee w$ is shared

(c) Another representation of $(x \vee y) \wedge (u \vee w)$

(d) A representation of $x \wedge \neg y \wedge z$

The vertical position of the variables corresponds to the order on them: the initial state is on the top and the leaves at the bottom. The False *transitions are depicted with dashed edges and* True *transitions with solid edges.*

Figure 2.5: Example BDDs

Section 2.3 gives more detail about SAT solvers. Nevertheless, most of the following text treats solvers as black-boxes that return either a model or a proof.

2.2.2 Binary Decision Diagrams (BDDs)

***B**inary **d**ecision **d**iagrams* (*BDDs*) are a compact representation of satisfying assignments of a certain formula [34].

The idea behind BDDs is to explicitly represent for each variable assignment whether the represented formula evaluates to True or False under that assignment. A BDD is an automaton with one initial state where the non-leaf nodes represent variables and the leaves the values for the whole formula. Each non-leaf node has two outgoing transitions, one for the value False of the variable and the other for the value True. To obtain the value of the formula under a certain assignment, one follows the appropriate edges until reaching one of the leaves.

Figure 2.5 presents several examples of BDDs. Figure 2.5a represents the formula $x \vee y$. Note that the only way how to reach the leaf F is by following the False transitions for both variables. This means that the formula evaluates to False if and only if both x and y are False.

We talk about a *reduced BDD* (*RBDD*), if no sub-BDD appears more than once in the BDD and no node makes both transitions to the same node.

Figure 2.5b represents the formula $(x \vee y) \wedge (u \vee w)$. Note that the sub-BDD corresponding to $u \vee w$ is shared by the True transitions coming out of x and y.

The reason why that is possible is that once $x \vee y$ is TRUE, whether the formula is TRUE or FALSE depends solely on u and w.

We talk about an *ordered BDD* (*OBDD*) if there exist a linear order on the variables such that no variable makes a transition to a variable that precedes it in that order. As illustrated by the following example, the order may affect the size of the BDD.

Figure 2.5c is another representation of $(x \vee y) \wedge (u \vee w)$ where different order of variables is used. Note that the size of this BDD is larger than the one in Figure 2.5b. Intuitively, the reason for that is that the the values of y and w are not independent of x and u.

In the BDD in Figure 2.5d the only way how to reach the leaf TRUE is to follow the TRUE transition from x, FALSE transition from y, and TRUE transition from z. This means that the BDD represents the formula $x \wedge \neg y \wedge z$.

We talk about a *ROBDD* if a BDD is both reduced and ordered. Note that all BDDs in Figure 2.5 are ROBDDs. Typically, the term BDD implicitly implies ROBDD as most implementations enforce order and reduction.

2.3 Better Understanding of a SAT Solver

This section explains the basic principles of modern SAT solvers. Understanding of this is necessary only for Section 4.4 and Section 6.3.1. This section greatly simplifies the workings of a SAT solver and the interested reader is referred to [138, 139, 158, 61].

Algorithms behind modern SAT solvers have evolved from the Davis-Putnam-Longeman-Loveland procedure (DPLL) [50]. The basic idea of DPLL is to search through the state space while performing constraint propagation to reduce the space to be searched. Indeed, a SAT solver searches through the space of possible variable assignments while performing unit propagation. Whenever the assignment violates the given formula, the solver must backtrack. The search terminates with success if all variables were assigned and the formula is satisfied. The search terminates unsuccessfully if the whole search space was covered and no satisfying assignment was found.

Before explaining the algorithm in greater detail, several terms are introduced. The solver assigns values to variables until all variables have a value. A value may be assigned either by propagation or by a *decision*.

Assignments made by propagation are necessitated by the previous assignments. Decisions determine the part of the search space to be explored. Assignments will be denoted by the corresponding literal, i.e., the assignment x means that the variable x is assigned the value TRUE, the assignment $\neg x$ means that the variable was assigned the value FALSE.

A state of the solver consists of a partial assignment to the variables and the *decision level*—the number of decisions on the backtracking stack. A *unit clause* is a clause with all literals but one having the value FALSE and the remaining one without a value. A *conflict* is a clause whose all literals have the value FALSE.

Whenever a unit clause is found, the unassigned literal must have the value TRUE in order to satisfy that clause. Whenever a conflict is found, the current variable assignment cannot satisfy the formula as all clauses must be satisfied. The following example illustrates these terms.

Example 11. *If the solver makes decisions x and y, the decision level is 2 and the clause $\neg x \vee \neg y \vee z$ is a unit clause. While for the decisions x, y, $\neg z$ the same clause is a conflict.*

Propagation consists of identifying unit clauses and assigning the value TRUE to the unassigned literal. Propagation terminates if a conflict is reached or there are no unit clauses left.

Example 12. *Let $\phi \stackrel{\text{def}}{=} (\neg x \vee y) \wedge (\neg x \vee \neg y)$ and the decision made is x. Both clauses are unit, if propagation picks the first one, it assigns to y the value* TRUE *and the second clause becomes a conflict. If, however, the solver makes the decision y, the propagation assigns to x the value* FALSE *as the second clause is unit. Note that in this case there is no conflict and we have a satisfying assignment for ϕ.*

Figure 2.6 shows pseudo-code for a SAT solver based on the techniques outlined so far. The search alternates between propagation and making decisions. Whenever the propagation detects a conflict, the search must backtrack, which means flipping the decision for the highest level where only one decision has been tried. This means, if the solver made the decision $\neg v$, it is flipped to v on backtracking and vice-versa.

The SAT solver described above has an important property. That is once the solver commits itself to a part of the search space, it finds a solution there if one exists. This is captured by the following property.

Property 1. *If the solver makes the decisions $l_1, \ldots, l_n$ and there exists a solution to the given formula satisfying these decisions, the solver returns such a solution.*

Example 13. *Let $\phi \stackrel{\text{def}}{=} (\neg x \vee \neg y \vee z) \wedge (\neg x \vee \neg y \vee \neg z)$. The propagation does not infer any new information and the solver makes the decision x. Again the propagation does not do anything and the solver makes the decision y. Now, the propagation infers a conflict and the solver must backtrack. The highest decision*

```
SAT
 1  level ← 0
 2  while TRUE
 3        do isConflict ← PROPAGATE
 4           if isConflict
 5              then backtrackLevel ←highest level where
                          only one decision has been tried
 6                   if such level does not exist
 7                      then return null
 8                      else backtrack to the level backtrackLevel
 9                           and flip the decision made there
10              else if all variables have a value
11                      then return current variable assignment
12                      else level ← level +1
13                           MAKEDECISION
```

Figure 2.6: Basic SAT solver pseudo-code

level where both decisions where not tried is 2 and thus the backtracking flips the decision y to the decision $\neg y$. Deciding z at the level 3, yields a satisfying assignment.

> ***Remark 2.5.*** *Several important techniques were omitted from the discussion above. In particular* clause learning *[138] which adds clauses to prevent conflicts.* Restarts *which restarts the solver if it is not reaching a solution for a long time [83]. And, heuristics used to pick the next decision [61].*

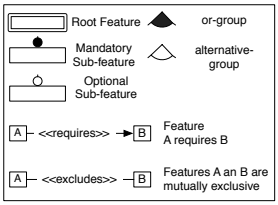
2.4 Variability Modeling with Feature Models

As mentioned in the introduction, in SPLE a program family targeted by a particular product line is modeled explicitly. Such models serve both the customer as well as the developer. For the customer, the model is a catalog of available products. For the developer, the model facilitates the reuse of the core assets by disallowing their wrong settings.

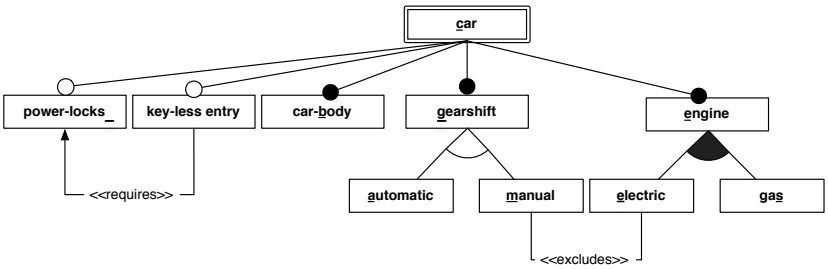
Feature models [114] are variability models widely accepted by the community. As the name suggests, the building block of the modeling is a *feature* defined by Kang et al. as follows.

> ... a prominent or distinctive user-visible aspect, quality, or characteristic of a software system or system. [114]

Using a certain set of features, an individual product is identified by a par-



(a) FODA modeling primitives



(b) Car feature tree

Figure 2.7: Example of a feature tree inspired by [48]

ticular combination of features that it supports. To relate to our discussion about problem and solution spaces (Figure 1.1), a designer of a product line needs to decide which potential products will be supported. Feature models enable capturing this set in terms of possible feature combinations.

- *Feature models describe valid products of a product line in terms of the feature combinations that are supported.*

2.4.1 Syntax for Feature Models

Even though conceptually a feature model corresponds to a set of valid products, it is uncommon to record a model as a list of these products. The reason is that explicit listing leads to large representations, and, humans prefer to represent sets in terms of dependencies.

A dependency can be for instance "the feature f_1 *requires* the feature f_2". If we assume that this is the only dependency, listing the corresponding set yields the combinations: no features ($\emptyset$), only f_2 ($\{f_2\}$), and both features ($\{f_1, f_2\}$). In the following text we sometimes talk about feature models as a set of permissible combinations and sometimes as a set of dependencies. The advantage of the combination-set is that it is unique while the advantage of the dependency-based representation is that it is compact. On a technical note, there is a large body of research concerned with knowledge representation, see e.g. [49].

- *Feature models are typically expressed as dependencies among features.*

2.4.2 FODA Notation

Kang et al. noticed that humans not only like to record sets of possibilities in terms of dependencies but also like to organize things hierarchically. Kang et al. introduced a user-friendly diagrammatic notation for recording features, commonly known as the *FODA notation*. As the notation is diagrammatic, it is best explained in figures.

Figure 2.7a shows the modeling primitives of the notation. Figure 2.7b, is an example of a *feature tree* representing possible configurations of a car. Each node in the tree represents a feature, the children of a node are called *sub-features* of the node. Some sub-features are grouped in order to express dependencies between siblings. The node at the very top is called the *root feature*.

In this example, the diagram expresses that a car must have a body, a gearshift, and an engine; an engine is electric *or* gas (selecting both corresponds to a hybrid engine); a gearshift is *either* automatic or manual. Additionally, a car may have power-locks or enable key-less entry. The two edges not part of the hierarchy are called *cross-cutting constraints*. In this example they express that a key-less entry cannot be realized without power-locks and the electric engine cannot be used together with manual gearshift.

In general, this notation might not suffice to nicely express some dependencies and then the designer can add dependencies in the form of an arbitrary logic formula; such formulas are called *left-over constraints* or *cross-tree constraints*.

> ***Remark 2.6.*** *Technically, left-over constraints are not needed as the FODA notation enables expressing any propositional formula [179]. In practice, however, left-over constraints are indispensable when the designer of the model needs to capture a dependency between features in different branches of the hierarchy. This is likely to occur as the hierarchy is not so much driven by the propositional semantics that the feature model is to capture but by the hierarchical organization of the features as seen in the real world by humans. For instance, gear-shift is a sub-feature of a car because gear-shift is a component of a car in the real world.*

Feature trees suggest a nice way of selecting a particular feature-combination (a product). The user selects or eliminates features starting from the root and continues toward the leaves. The user must follow a simple set of rules in order to stay within valid products. If the user has selected a feature, then he or she must select a sub-feature if that is a *mandatory feature*; that is not required if it is an *optional sub-feature*. If the user selects a feature then *one and only one* sub-feature must be selected from an *alternative-group*[1] and *at least* one must

[1] Sometimes alternative-groups are called *xor-groups*, which is misleading for groups with

modeling primitive	**logic formula**
Root(r)	v_r
OptionalChild(c, p)	$v_c \Rightarrow v_p$
MandatoryChild(c, p)	$v_c \Leftrightarrow v_p$
Excludes(a, b)	$\neg(v_a \wedge v_b)$
Requires(a, b)	$v_a \Rightarrow v_b$
AlternativeGroup(p, $\{f_1, \ldots, f_n\}$)	$(v_{f_1} \vee \cdots \vee v_{f_n} \Leftrightarrow v_p) \wedge \bigwedge_{i<j} \neg(f_i \wedge f_j)$
OrGroup(p, $\{f_1, \ldots, f_n\}$)	$v_{f_1} \vee \cdots \vee v_{f_n} \Leftrightarrow v_f$
Constraint(ϕ)	ϕ

Table 2.1: Semantics of modeling primitives

be selected from an *or-group*.

If a feature f_1 `requires` the feature f_2, once f_1 is selected f_2 must be as well. Features connected with an `excludes` edge cannot be both selected. Chapter 3 discusses this process in greater depth and the precise semantics of feature models is discussed formally in the upcoming section.

- *FODA notation is a user-friendly diagrammatic notation for feature models which organizes the features in a hierarchy.*

Extensions of FODA Notation

The original FODA notation gave raise to several extensions. A natural generalization of *xor* and *or* groups is the *cardinality-based notation* [47, 46, 43]. The cardinalities on a feature-group express the minimum and maximum of features that must be selected whenever the parent-feature is selected as in Entity-Relation or UML modeling.

Then there are some extensions that take feature models out of the realm of finiteness. *Feature attributes* (also called *properties*) enable expressing variability with large or infinite domain, such as numeric values or strings. Often, an attribute is fixed, e.g., a price of a feature.

Feature cloning [47] enables the user to require several copies of a feature. These copies are typically distinguished by an attribute, as it is not clear what it means to have the same exact feature multiple times.

- *The FODA notation has several extensions; some of these enable expressing variability with non-Boolean or infinite domains.*

2.4.3 Semantics of Feature Models

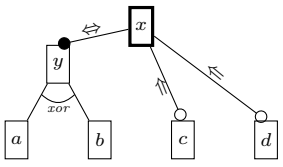
Diagrammatic notations for feature models are user-friendly but not machine-friendly. Namely, if we want to reuse tools for automated reasoning—such as

more than two elements.

SAT solvers—we have to provide a semantics in terms of mathematical logic. Additionally, a mathematical semantics increases the confidence that everybody understands the syntax in the same way.

Propositional logic has proven to be a good candidate for FODA semantics [179]. The semantics consists of a set of variables and a formula on these variables. For each feature f, the variable v_f is introduced representing whether the pertaining feature is selected or not. The formula is constructed such that there is a one-to-one correspondence between satisfying assignments of the formula and the permissible feature combinations of the feature model.

Table 2.1 presents the semantics of the individual modeling primitives of the FODA notation; the semantics of a feature model is a conjunction of the individual semantics. Note that there is always an implication from a sub-feature to its parent, i.e., a sub-feature cannot be selected without its parent.



(a) Feature model example

$$
\begin{aligned}
& v_x \wedge \\
& v_y \Leftrightarrow v_x \wedge \\
& v_c \Rightarrow v_x \wedge v_d \Rightarrow v_x \wedge \\
& v_a \Rightarrow v_y \wedge v_b \Rightarrow v_y \wedge \\
& v_y \Rightarrow (v_a \vee v_b) \wedge \\
& v_y \Rightarrow \neg(v_a \wedge v_b)
\end{aligned}
$$

(b) Its semantics

Figure 2.8: FODA notation and its semantics

Figure 2.8 illustrates the semantics on a concrete toy-example. The left-hand side of the figure is a small feature tree in the FODA notation[2]. The right-hand side captures its semantics in term of a propositional formula obtained by using the rules from Table 2.1.

Semantics of Extensions of FODA

As most of this thesis is dealing with the actual logic semantics of feature models rather than the syntax, we are not going into great detail of semantics of FODA extensions and the reader is referred to relevant publications. Propositional logic semantics have been extensively studied by Schobbens et al. [179].

Jonge and Visser [51] propose the use of context-free grammars for capturing the semantics of feature models. Feature configurations conforming to a feature model are defined as the sentences generated by the pertaining grammar. This idea is extended by Batory [15] by combining grammars and propositional logic.

Czarnecki et al. [47] also use context-free grammars to capture the semantics of feature models. Their meta-model is more expressive than the feature models

[2]The labels on the edges are not part of the notation and are meant to represent the semantics.

covered by the aforementioned articles, as their meta-model enables *cardinalities* and *feature cloning*. The main limitation of grammars is that they do not enable cross-tree dependencies.

Höfner et al. take an algebraic approach to feature modeling. In this approach, a product family is a set of sets of features, and the language of algebra is used to abstract from the sets. Dependencies, such as exclusion, are expressed as additional equalities. What makes this approach interesting is that it enables to use multisets (sets with repetitions of an element, also known as *bags*) instead of sets. This extension permits the modeling of feature models with cloning. Even though this approach seems interesting, no other publications along this line of research are known [97].

Benavides et al. user Constraint Satisfaction Problems to capture semantics of feature models with attributes [21].

The author of this dissertation developed an approach to formalizing feature models in higher-order logic using the proof assistant PVS. Thanks to the expressive power of the PVS language and higher order logic, this approach enables reasoning about the full range of feature models—with cloning or attributes [101].

Chapter 3

Configuring Variability Models

We established that variability models represent a set of configurations that a particular product line is set out to be able to implement. If users are configuring such model, they are looking for a particular configuration that respects the model and also fulfills their requirements. In this chapter we establish a connection between configuration and propositional logic.

In the case of feature models, the set of configurations is not recorded as a catalog of configurations but as a set of constraints imposed on these configurations since constraints enable more compact representation. However, it is hard for a user to keep track of all the constraints for large and complex systems. Hence, it is practical for the users to specify their requirements on the desired configuration in a series of steps and reflect on the result of each step.

Section 2.4.1 suggests that for the FODA notation it is intuitive to start making decisions about features from the root towards the leaves. In the following, however, we do not assume any particular order of selecting or eliminating features as that would be an unnecessary assumption about the user.

In fact, in general there is little that we assume about the decisions that users might make. We do assume, however, that they wish to respect the constraints of the product line specified by the feature model. This assumption is justified by the fact that breaking the constraints given by the feature model would likely require a change in the whole product line and thus incur higher development cost, or, it might be the case that the constraint represents an actual physical impossibility.

Conceptually, the configuration process can be thought of as a series of sets of configurations (see Figure 3.1). The first set comprises the set of configurations

permitted by the variability model (e.g., all permissible feature combinations), further decisions put by the user shrink this set until the set contains exactly one element (e.g., a particular feature combination). In this sense, a configuration process is a step-wise refinement [200, 47, 46, 106].

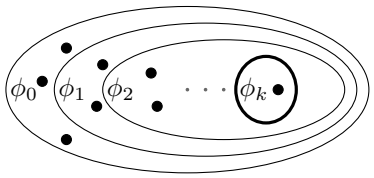


Figure 3.1: Configuration schematically

- *The objective of a configuration process is to find a configuration conforming both to the constraints given by the configured variability model and the requirements specified by the customer.*

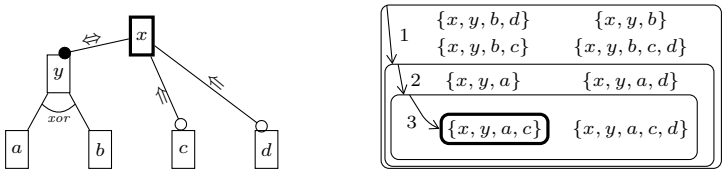
The purpose of this chapter is to set the ground for *interactive configuration*, a configuration process during which a tool provides some feedback after each decision or on request. The word 'interactive" stems from the fact that the process is not just a set of decisions laid out by the customer and then checked for validity, but, it involves responses from the tool which helps the customer stay within validity bounds.

- *An interactive configuration process is a sequence of steps where a tool provides support throughout the process.*

3.1 Propositional Configuration

As we saw in Section 2.4.3, propositional logic is used to express a semantics of the FODA notation and some of its extensions. A configuration process for a FODA feature model is exemplified by Figure 3.2b, in accordance with the principles outlined above (compare to Figure 3.1). In the first step the user selects the feature a, which discards all the feature configurations not containing it and effectively those that contain the feature b, as a and b are alternatives. In the second step the user selects the feature c which discards all the feature configurations not containing it. In the last step the user eliminates the feature d which discards configurations that contain it and which leaves us with a single feature configuration that conforms to the feature model and the user decisions made.

The example illustrates that a configuration process of a FODA feature model can be seen as a gradual refinement of the set of permissible configurations. In practice, however, we do not represent these sets explicitly but represent them as formulas.



(a) Feature model example

(b) Example configuration of Figure 3.2a

Figure 3.2: FODA notation and configuration processes

This section defines terms and concepts for configuration of variability models with propositional semantics in general. Hence, further on we do not refer to features and feature models, rather, we assume that they have been translated into the corresponding variables and formulas (see Section 2.4.3). The discussion on how to maintain a link between variability models and their semantics is left to Chapter 7.

Interactive configuration consists of *user actions* and responses from the configurator. This dissertation focuses on the following user actions: requests for *explanations*, *user decisions* (or just decisions), and *retractions*. A user decision is either an elimination or selection of a variable. Retracting a decision means bringing the configuration process into a state as if that decision had not been made. Explanation is a justification why the configurator has disabled a certain decision from being made.

In terms of formulas, selecting a variable v corresponds to the formula v and eliminating v corresponds to the formula $\neg v$. Note that decisions are literals (see Section 2.1.3). In propositional logic, a literal and a value of a variable carry the same information: the literal x corresponds to x being TRUE, the literal $\neg x$ corresponds to x being FALSE. The following text uses these formalisms interchangeably.

A configuration process starts from some initial formula that is to be configured; this formula might correspond to a feature model, for instance. Further, the user is making decisions or retracting some of the old ones. Formally we model the process as a sequence of sets of decisions that are active in each step.

Definition 2. *A propositional configuration process for an initial formula ψ, where ϕ is satisfiable, is a sequence of sets of user decisions $D_0, \ldots, D_k$ with the following properties.*

The initial set of decisions is empty ($D_0 = \emptyset$). For $i > 0$, the set of decisions is obtained from the previous set either by adding or removing a decision, i.e., either $D_{i+1} = D_i \cup \{d\}$ or $D_{i+1} = D_i \setminus \{d\}$ where d is a literal.

The indices in the sequence are called steps. The state formula in the step

$i \in 0 \dots k$ *is the formula* $\psi_0 \wedge \bigwedge_{d \in D_i} d$.

Recall that the goal of a configuration process is to find a particular solution of the initial formula. Hence, we consider a configuration process to be complete if it determines one particular solution. In the context of propositional logic, solutions correspond to satisfying variable assignments (models of the formula). These concepts are captured in the following definition.

Definition 3. *We say that a configuration process is complete iff the state formula in the last step has one and only one satisfying assignment.*

A configurator is a tool that assists the user during a configuration process. The main role of a configurator is to prevent the user from making "wrong" decisions. Informally, a decision is wrong if the configuration cannot be completed without changing this decision. On the other hand, it should not prevent the user from finding permissible solutions. The following definition captures these two requirements.

Definition 4. *We say that a configurator is backtrack-free iff does not allow configuration processes with unsatisfiable state formulas.*

A configurator is complete iff for any satisfying assignment of the initial formula there exists a complete process where the state formula in the last step is satisfied by this assignment.

> ***Remark 3.7.*** *There does not seem to be any motivation on behalf of the user for a configurator to be not complete. If a configurator is not complete, it means that the designer of the instance being configured included some possibilities that the user cannot reach. However, Freuder shows that the problem of backtrack-free interactive configuration can become computationally easier if completeness is sacrificed [77].*
>
> *In some approaches, the user is enabled to temporally choose non-backtrack-free options. This might happen if some external conditions change or if there are multiple users on figuring the same instance. In these cases interactive reconfiguration or corrective explanations can be used to get back within the overall constraints [91, 159, 199]. However, the configurator needs to be aware of the fact that backtrack-freeness was violated in order to invoke such mechanism.*

In the remainder of this section we make several observations about the properties of the definitions above. These properties are closely tied to the concept of bound variables and literals, which are defined as follows.

Definition 5. *We say that a variable v is* bound to the value TRUE in a formula ϕ *iff it has the value* TRUE *in all satisfying assignments of ϕ. Analogously, we say that a variable v is* bound to the value FALSE in a formula ϕ *iff it has the value* FALSE *in all satisfying assignments of ϕ. We say that a variable v is bound if it is bound to some value. In a mathematical notation, for v bound to* TRUE *it holds $\phi \models v$ and it holds $\phi \models \neg v$ for v bound to* FALSE.

We extend the terminology for literals as well, we say that a literal l is bound in ϕ *iff it evaluates to* TRUE *in all satisfying assignments of ϕ, i.e., it holds $\phi \models l$.*

Note that a variable v is bound in a formula iff one of the literals v, $\neg v$ is bound in the formula. If both of these literals are bound, the formula is unsatisfiable. To determine whether a literal is bound, it is possible to ask the question whether the complementary literal can ever be satisfied; this is formalized by the following proposition. (Recall that $\bar{l}$ denotes the literal complementary to the literal l, Section 2.1.3.)

Proposition 1. *A literal l is bound in ψ iff the formula $\psi \wedge \bar{l}$ is unsatisfiable.*

Proof. First we observe that $\psi \wedge \bar{l}$ has those satisfying assignments that satisfy both ψ and $\bar{l}$. Additionally, for any assignment either l or $\bar{l}$ holds (but not both).

If $\psi \models l$, then all satisfying assignments of ψ satisfy l and therefore there are no assignments satisfying $\bar{l}$. If there are no satisfying assignments of $\psi \wedge \bar{l}$, then l must be satisfied by all satisfying assignments of ψ. □

The following observation relates bound variables and complete processes.

Observation 1. *A configuration process is complete iff all variables are bound in the state formula of the last step.*

Example 14. *Let ϕ_i denote state formulas of a process starting from $\phi_0 \stackrel{\text{def}}{=} (\neg u \vee \neg v) \wedge (x \Rightarrow y)$. The user selects u ($\phi_1 \stackrel{\text{def}}{=} \phi_0 \wedge u$); v is bound to* FALSE *as u and v are mutually exclusive. The user sets y to* FALSE *($\phi_2 \stackrel{\text{def}}{=} \phi_1 \wedge \neg y$); the variable x is bound to* FALSE. *The process is complete as all variables are bound.*

The following propositions show that a backtrack-free and complete configurator enforces decisions that are bound. Conversely, it disallows decisions whose complement is bound. This observation is used in Chapter 4 to construct a configurator.

Proposition 2. *A configurator is backtrack-free iff it does not permit decisions d such that $\bar{d}$ is bound in the current state formula.*

Proof. In the following, ϕ_i and ϕ_{i+1} denote two consequent state formulas. First, if a configurator is backtrack-free, then it must not allow to make a decision d if

$\bar{d}$ is bound. Since if it did allow such decision, it holds $\phi_i \models \bar{d}$ and $\phi_{i+1} \equiv \phi_i \wedge d$ where ϕ_{i+1} is unsatisfiable due to Proposition 1.

Now we show, by contradiction, that if it does not allow to make decisions d whose complement is not bound, then the configurator is backtrack-free. Let us assume that the configurator is not backtrack-free and therefore there exists an unsatisfiable state formula ϕ_i in some process. Since the initial formula is satisfiable by definition, the first state formula cannot be the unsatisfiable one. If a state formula $\phi_i \equiv \phi_{i-1} \wedge l$ is unsatisfiable, then $\phi_{i-1} \models \bar{l}$ due to Proposition 1 and that is a contradiction because the decision l is not allowed. □

Proposition 3 (Completeness). *If a configurator allows any decision whose complement is not bound, then it is complete.*

Proof. Let $l_0, \ldots, l_n$ be literals representing a satisfying assignment of ϕ_0, in some (arbitrary) order. Since $\phi_0 \wedge l_0 \wedge \cdots \wedge l_i \wedge l_{i+1}$ is satisfiable for any $i \in 0..n-1$, the literal $\bar{l}_{i+1}$ is not bound due to Proposition 1. Hence, the configurator allows the decision l_{i+1} in the step i. The desired assignment is obtained by the decisions $l_0, \ldots, l_n$ in this order. □

3.2 Summary

A configuration process can be seen as a gradual refinement of the set of configurations. In SPLE we represent the set of configurations using feature models.

Feature models captured in the FODA notation have semantics expressible in propositional logic. We translate the given feature model into its semantics and similarly represent user decisions as logic formulas. Conjoining the semantics of a feature model with user decisions gives us the *state formula* (Definition 2).

A *configurator* is a tool that provides a user with feedback throughout the configuration process (hence making the process *interactive*). A configurator is *backtrack-free* iff it does not allow such sequences of user decisions that cannot be extended into a valid configuration. A configurator is *complete* iff it allows all sequences of user decisions that lead into a valid configuration (Definition 4).

We define the term *bound literal* (Definition 5) as a literal that is true in all valid configurations, and, we show that whether a literal is bound or not, can be phrased as a satisfiability question (Proposition 1). Finally, we show that a completeness and backtrack-freeness of a configurator is achieved by identifying bound literals throughout the process. This is particularly important in the following chapter as it makes a connection between backtrack-free and complete configurators, and, satisfiability.

Chapter 4

Interactive Configuration with a SAT Solver

This chapter describes how to use a SAT solver to implement a complete and backtrack-free configurator. The implementation relies on propositions 2 and 3 which tell us that completeness and backtrack-freeness is achieved by identifying bound literals.

The presentation is carried out in the following steps. Section 4.1 describes the context in which the algorithms operate. Section 4.2 presents the main ideas of an algorithm for finding bound literals and a naive version of this algorithm. This naive version is improved in Section 4.3; this improvement has the property of being easy to implement while it significantly improves the performance. Section 4.5 focuses on an aspect neglected by the previous sections—the explanations for the feedback given to the user by the configurator.

4.1 Setup

All the presented algorithms expect the initial formula to be given in the conjunctive normal form (CNF); how the CNF is obtained is discussed in Chapter 7. The presentation of the algorithms is done in the context depicted in Figure 4.1. The centerpiece is the configurator which communicates with two components: the *user interface* and a *SAT solver*.

The communication is carried out through a set of messages. The user interface initializes the configurator with an initial CNF formula by sending the message INIT(ψ : *CNF*); all the inference from then on assumes that this formula holds and therefore the message requires that the formula is satisfiable. The user interface communicates a new user decision by the message ASSERT(l : *Literal*).

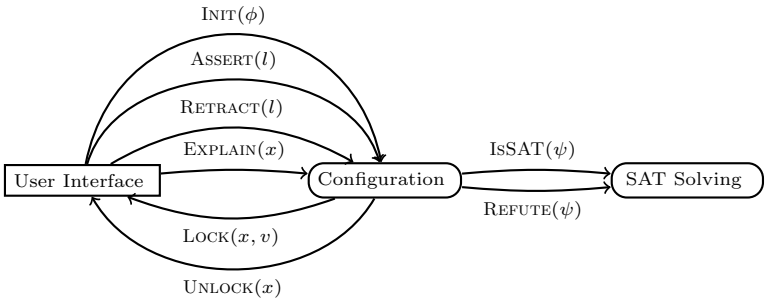


Figure 4.1: Configuration process

The user interface requests to retract a previously made decision by sending Retract(l : *Literal*). Asserting a previously asserted literal without retracting it has no effect. Analogously, retracting a literal not asserted has no effect. Asserting the literal v corresponds to selecting the variable v and asserting the literal $\neg v$ corresponds to eliminating the variable. The user interface may send Assert and Retract messages only with literals that contain a variable that appears in the initial formula ψ sent in the initialization message. More specifically, this means that the set of variables whose values are to be configured remains the same throughout the configuration process.

> ***Remark 4.8.*** *The actual implementation enables configuring variables not appearing in ψ_0 by letting the user interface specify explicitly the set of variables that should be considered. However, these variables are trivially unbound independently of the other variables and can become bound only if the user interface sends the corresponding* Assert *message.*

The configurator uses the message Lock(x : *Variable*, v : *Boolean*) to lock a variable in a certain value and it uses Unlock(x : *Variable*) to unlock a variable. Note that the pair (x : *Variable*, v : *Boolean*) passed to the message Lock corresponds to a literal: if v is True, the corresponding literal is x; if v is False, the corresponding literal is $\neg x$.

The user interface guarantees that it does not send Assert messages concerning a locked variable. In other words, the user interface prevents its users from changing values of locked variables. The message Lock(x : *Variable*, v : *Boolean*) is sent when the variable x is bound to the value v.

Note that the mechanism of locking ensures completeness and backtrack-freeness of the configurator due to propositions 3 and 2. If one of the literals v, $\neg v$ is bound, the configurator locks the variable v and therefore the user can never make a decision whose complement is bound.

If a certain variable is locked, the user interface may ask *why* it has been locked by sending the message Explain(x : *Variable*). The precondition for calling Explain is that the variable x is locked in some value v by Lock. The return value of Explain is a resolution-based proof of why x must have the value v (this message is discussed in greater detail in Section 4.5).

The configurator queries the SAT solver by the message IsSAT(ψ : *CNF*) whose return value is either **null** iff ψ is unsatisfiable, or, a set of literals that corresponds to a satisfying assignment of ψ. For example, for the query IsSAT($x \vee y$) the possible responses are $\{x, \neg y\}$, $\{\neg x, y\}$, and $\{x, y\}$. If a certain formula is unsatisfiable (the solver returned **null**), then the configurator may ask for a proof of the unsatisfiability by the query Refute(ϕ), this returns a resolution tree deriving $\bot$ from ϕ (see Section 2.1.3 for more details).

As discussed in Section 2.2, some SAT solvers do not provide a resolution-based proofs. Section 7.1 discusses how to deal with such obstacle. For the purpose of this chapter, we assume that the SAT solver produces a resolution tree as a proof.

> ***Remark 4.9.*** *In practice, a SAT solver may take too much time to respond. Here, however, we assume that the solver always responds. An actual implementation should have a safety mechanism that checks at the initialization whether the formula is unwieldy or not. This can be for instance achieved by calling the SAT solver on the initial formula upon start. If this first call times out, further calls will very likely timeout as well. On the other hand, if this call succeeds, then the further calls will likely succeed as well.*

4.2 Configurator Skeleton

To keep track of decisions made so far, the configurator maintains the set D representing the current user decisions as a set of literals. It stores the initial formula in the variable ϕ_0.

```
▷ requires
    ψ is satisfiable
Init(ψ : CNF)
    φ_0 ← ψ
    D ← ∅
    Test-Vars()
```

(a) Initializing

```
▷ requires
    l̄ is not locked
Assert(l : Literal)
    D ← D ∪ {l}
    Test-Vars()
```

(b) Asserting

```
Retract(l : Literal)
    D ← D ∖ {l}
    Test-Vars()
```

(c) Retracting

Figure 4.2: Basic implementation of Init, Assert, and Retract

```
TEST-VARS()
1  φ ← φ0 ∧ ⋀_{l∈D} l
2  foreach x : vars-of(φ0)
3        do CanBeTrue ← TEST-SAT(φ, x)
4           CanBeFalse ← TEST-SAT(φ, ¬x)
5           if ¬ CanBeTrue ∧ ¬ CanBeFalse
6              then error "Unsatisfiable"
7           if ¬ CanBeTrue then LOCK(x, FALSE)
8           if ¬ CanBeFalse then LOCK(x, TRUE)
9           if CanBeTrue ∧ CanBeFalse then UNLOCK(x)
```

Figure 4.3: Skeleton of the algorithm for testing bound variables

Figure 4.2 shows an implementation of the procedures INIT, ASSERT, and RETRACT. All of these procedures rely on the procedure TEST-VARS, whose job is to analyze the newly established state and compute the appropriate binding for the variables.

Figure 4.3 presents a pseudo-code for TEST-VARS which computes, for each literal, if it is bound in the current state formula. To do that, it relies on Proposition 1, which tells us that a literal l is bound in the formula ϕ iff $\phi \wedge \bar{l}$ is unsatisfiable. In order to test this, the procedure TEST-VARS uses a function TEST-SAT($\psi : \mathit{CNF}, l : \mathit{literal}$), which tests the satisfiability of $\psi \wedge l$ and is discussed later on.

The two calls to TEST-SAT have four possible outcomes and these are investigated in the lines 5–9. If the variable x can be neither TRUE nor FALSE, then ϕ itself is unsatisfiable (line 5), which breaks the assumptions that ϕ is satisfiable at all times as the configurator is backtrack-free (this possibility occurs only when some of the contracts of the procedures are broken). If the variable can be *either* TRUE or FALSE, but not both, then the variable is locked in the complementary value (lines 7 and 8). If the variable can be both TRUE and FALSE, the algorithm makes sure that the variable is unlocked (line 9).

```
TEST-SAT(ψ: Formula, l: Literal) : Boolean
   return ISSAT(ψ ∧ l) ≠ null
```

Figure 4.4: Naive version of TEST-SAT

Now, let us look at the implementation of TEST-SAT. Since the algorithm has a SAT solver at its disposal, it can simply query the solver for the satisfiability of $\phi \wedge l$ as shown in Figure 4.4. This yields an algorithm that calls the solver

twice on each variable after each assertion and retraction. (TEST-SAT always invokes the solver and TEST-VARS invokes TEST-SAT twice for each variable.)

> ***Remark 4.10.*** *Note that the order of iterating through the set of variables in* TEST-VARS *does not influence which variables will be locked and which not. The reason for that is that whether a variable v is locked or not solely depends on the satisfiability of $\phi \wedge v$ and $\phi \wedge \neg v$ and the formula ϕ does not change during the iteration.*

4.3 Computation Reuse

As pointed out, the version of TEST-SAT above calls the SAT solver considerably many times—twice for each variable after each user action. This is improved if we realize that the iterations of the loop in TEST-VARS do not in any way reuse information gained in the previous iterations. Let us look how that can be done.

First, consider the situation when the algorithm tests the satisfiability of $\phi \wedge l$ and the solver returns a satisfying assignment. If that assignment gives the value TRUE to some variable x, then $\phi \wedge x$ must be satisfiable as this assignment is a witness for the fact that $\phi \wedge l \wedge x$ is satisfiable. Therefore, there is no need to call the solver on $\phi \wedge x$ after that. Analogously, if the assignment gives FALSE to x, then $\phi \wedge \neg x$ is satisfiable and there is no need to call the solver on $\phi \wedge \neg x$.

Example 15. *Let $\psi \stackrel{\text{def}}{=} x \vee y \vee z$ and assume that* IsSAT$(\psi \wedge x)$ *responds $\{x, y, \neg z\}$. From the response of the solver we know that $\psi \wedge x$ is satisfiable but also that $\psi \wedge y$ and $\psi \wedge \neg z$ are satisfiable.*

Based on this observation, the first improvement to the algorithm is to avoid satisfiability tests of values that appear in some of the satisfying assignments encountered so far. This optimization does not exploit, however, the negative responses of the solver (**null**). Consider the situation when the constraint contains the formula $\neg x_1 \vee x_2$. If the solver knows that x_2 must be FALSE, it can quickly deduce that x_1 must be FALSE by using unit propagation (see Section 2.1). In general, conjoining the bound literal gives to the solver more information about the formula and the solver is expected to perform better. This motivates the second improvement: storing the values that are disallowed and conjoining their complements to the whole formula in subsequent queries. We refer to this optimization as *catalyst optimization* as the literals being conjoined should speed

up the solving process (serve as catalysts).

> ***Remark 4.11.*** *The optimization just introduced is analogous to the way lemmas are used in mathematical proofs. If $\phi \wedge l$ is unsatisfiable, then $\phi \models \bar{l}$ and this knowledge is used in further inference just as lemmas are used to prove other theorems.*

Based on these two ideas, we devise a new version of Test-SAT. This version is again called from Test-Vars (see Figure 4.3).

```
Test-SAT(ψ: CNF, l: Literal) : Boolean
1  if l ∈ KnownValues then return True
2  if l ∈ DisabledValues then return False
3  L ← IsSAT(ψ ∧ l ∧ ⋀_{k∈DisabledValues} k̄)
4  if L ≠ null
5     then KnownValues ← KnownValues ∪ L
6     else DisabledValues ← DisabledValues ∪ {l}
7  return L ≠ null
```

Figure 4.5: Improved Test-SAT

This new version (see Figure 4.5) relies on two sets *DisabledValues* and *KnownValues*. The set *DisabledValues* comprises literals l for which the algorithm knows that $\phi \wedge l$ is *unsatisfiable*, and the set *KnownValues* comprises the literals l for which the algorithm knows that $\phi \wedge l$ is *satisfiable* for the current state formula ϕ. For instance, if *DisabledValues* contains the literal $\neg x$, then x must not be False in the current state formula—it is bound to True; if *KnownValues* contains the literal $\neg y$, then y can be True in the current state formula so it is certainly *not* bound to False.

How the sets *DisabledValues* and *KnownValues* are updated between different calls to Test-Vars is discussed after the description of Test-SAT—recall that Test-SAT is invoked from Test-Vars on the current state formula.

Test-SAT first consults the two sets and does *not* invoke the solver if the queried literal is in any of them (lines 1 and 2). If it *does* call the solver and $\psi \wedge l$ is satisfiable, it adds into *KnownValues* all the literals that are in the returned satisfying assignment (line 7). If $\psi \wedge l$ is unsatisfiable, then it adds the literal l into the set *DisabledValues* (line 8).

Note that if a variable x is not bound to any value, then eventually both literals x and $\neg x$ will be in the set *KnownValues*. In contrast, the literals x and $\neg x$ will never both end up in the set *DisabledValues* as that would mean that x can be neither True nor False. Hence, if a literal l is in *DisabledValues*, then the literal $\bar{l}$ will end up in *KnownValues* as the literal $\bar{l}$ is bound in the current

```
INIT(ψ : CNF)
    φ ← ψ
    D ← ∅
    KnownValues ← ∅
    DisabledValues ← ∅
    TEST-VARS()

ASSERT(l : Literal)
    D ← D ∪ {l}
    KnownValues ← ∅
    TEST-VARS()

RETRACT(l : Literal)
    D ← D ∖ {l}
    DisabledValues ← ∅
    TEST-VARS()
```

Figure 4.6: Change of state and *KnownValues* and *DisabledValues*.

state formula.

Observation 2. *If $\psi \wedge l$ is satisfiable, then the literal l itself must be among the returned literals. Therefore* TEST-SAT *never invokes the solver twice on the same query.*

So far it has not been specified how to initialize the sets *KnownValues* and *DisabledValues* neither it was said how asserting and retracting affects them. A rather coarse approach would be to empty both sets whenever ϕ changes, i.e., in INIT, ASSERT, and RETRACT. However, any assertion strengthens ϕ and therefore it is safe to keep *DisabledValues*. Conversely, decision retracting weakens ϕ and it is safe to keep *KnownValues*. This is captured by the two following observations.

Observation 3. *After a retraction, the literals in the set KnownValues are still possible assertions for the new state-formula.*

Proof. Let $\phi \equiv \phi' \wedge l$ be the state-formula before the retraction and ϕ' be the state-formula after retraction. The formula ϕ' "constrains less" (it is weaker). In other words, all the satisfying assignments of ϕ are also satisfying assignments of ϕ'. Hence, if $\phi' \wedge l \wedge k$ is satisfiable for some literal $k \in \mathit{KnownValues}$, then $\phi' \wedge k$ is also satisfiable. □

Observation 4. *After an assertion, the literals in the set DisabledValues are still impossible to be asserted for the new state-formula.*

Proof. Let ϕ be the state-formula before the assertion and $\phi' \equiv \phi \wedge l$ the state-formula after the assertion. The formula ϕ' "constrains more" (it is stronger). Hence, if $\phi \wedge k$ is unsatisfiable for some literal $k \in \mathit{DisabledValues}$, then $\phi \wedge l \wedge k$ must be unsatisfiable as well. □

4.3.1 Example Execution

The following text describes a sample configuration process using the described algorithm. The notation ϕ_i is used to represent the state formula in step i. The

user interface initializes the process with the formula to be configured defined as follows.

$$\begin{aligned}\psi_0 \quad &\stackrel{\text{def}}{=} \quad p \Rightarrow (q \Rightarrow r) \wedge \\ &\qquad\quad p \Rightarrow (\neg q \vee \neg r)\end{aligned}$$

Initialization The procedure INIT stores the initializing formula ψ into ϕ_0 and sets each of the D, *DisabledValues*, and *KnownValues* to the empty set and calls TEST-VARS.

Initial Test The procedure TEST-VARS begins by testing the variable p. Since $\phi_0 \wedge p$ is satisfiable, the SAT solver finds a satisfying assignment $\{p, \neg q, r\}$, whose literals are added to *KnownValues*. Similarly, the query $\phi_0 \wedge \neg p$ yields the literals $\{\neg p, q, \neg r\}$, which also are added to *KnownValues*.

Thanks to the luck we had in finding satisfying assignments, the tests for the variables r and q do not need to use the SAT solver since all four pertaining literals are already in *KnownValues*.

Assert Now the user makes the decision that the variable p should be TRUE, this yields the invocation ASSERT(p). ASSERT adds the literal p to the set D and clears the set *KnownValues*. This gives us the state formula $\phi_1 \stackrel{\text{def}}{=} \phi_0 \wedge p$. For better intuition, the formula ϕ_1 can be rewritten as follows.

$$\begin{aligned}\phi_1 \quad &\stackrel{\text{def}}{=} \quad (q \Rightarrow r) \wedge \\ &\qquad\quad (\neg q \vee \neg r)\end{aligned}$$

Test after decision The procedure ASSERT invokes TEST-VARS. The SAT solver responds $\{p, r, \neg q\}$ and $\{p, \neg r, \neg q\}$ to the queries $\phi_1 \wedge r$ and $\phi_1 \wedge \neg r$, respectively. Hence, both TRUE and FALSE are possible for the variable r, while for q we only know that it can be FALSE under this state formula.

The SAT solver tells us that $\phi_1 \wedge q$ is unsatisfiable and the algorithm locks the variable q in the value FALSE. The configuration process is complete as all variables are bound.

> ***Remark 4.12.*** *Even this small example shows that unit propagation (see Section 2.1.3) is an insufficient inference mechanism for configuration (as in [15]). Applying unit propagation to* $p \wedge (p \Rightarrow (q \Rightarrow r)) \wedge (p \Rightarrow (\neg q \vee \neg r))$ *yields* $p \wedge (q \Rightarrow r) \wedge (\neg q \vee \neg r)$*, from which it is not obvious that* q *must be* FALSE.

4.3.2 Discussion about Complexity

This section makes several observations regarding the computational complexity of the approach. A first thing that should be justified is the use of a SAT solver. Satisfiability in general is an NP-complete problem. Couldn't determining the satisfiability of $\phi \wedge l$ be easier? The following claim shows that it is just as difficult.

Claim 1. *Let ϕ be a CNF formula and let l be a literal. Then the satisfiability of $\phi \wedge l$ (which is also CNF) is NP-complete.*

Proof. Let ψ be a formula and x a variable not appearing in ψ. Let ψ' be a CNF of the formula $x \Rightarrow \psi$. The formula ψ' is obtained from ψ by adding the literal $\neg x$ to each clause (hence, the size of ψ' is linear in the size of ψ). Since $\psi' \wedge x$ and ψ are equisatisfiable, answering the query of satisfiability of $\psi' \wedge x$ answers satisfiability of ψ, which means solving an NP-complete problem and thus the satisfiability of $\phi \wedge l$ is NP-hard. The satisfiability of $\phi \wedge l$ is in NP because it can be checked in polynomial time that an assignment satisfies the formula $\phi \wedge l$, i.e., the problem is NP-complete. □

Claim 2. *To determine whether a literal l is bound in a formula ϕ is co-NP complete.*

Proof. An immediate consequence of previous claim and Proposition 1. □

As the above claims show, the procedure Test-Vars (Figure 4.3) solves two NP-complete problems for each variable: the satisfiability of $\phi \wedge x$ and the satisfiability of $\phi \wedge \neg x$. As the SAT solver is the place where this happens, it is interesting to look at how many times the solver is called in that procedure.

Claim 3. *If n is the number of variables of the problem being configured, then the SAT solver is called no more than $n + 1$ times.*

Proof. For each variable y there are the two complementary literals y and $\neg y$. Therefore there are $2n$ different literals whose satisfiability the procedure Test-Vars must answer.

Let x be the first variable being tested in Test-Vars. At least one of the two calls to the solver for satisfiability of $\phi \wedge x$ and $\phi \wedge \neg x$ must return a satisfying assignment otherwise ϕ would be unsatisfiable. Therefore, after the first test of the first variable, the set *KnownValues* contains at least $n - 1$ literals that do not contain the variable x. In other words, we know that at least one of the literals corresponding to any of the variables succeeding x is satisfiable. □

Remark 4.13. *Here I would like to make a highly intuitive remark concerning the case when the solver is called many times (close to $n+1$). One scenario when the SAT solver is called $n+1$ times is when there is exactly one satisfying assignment of the formula. In that case the solver will respond negatively for all the literals that have not been tested yet after the test of the first variable. If that is the case, then it is likely that the variables are strongly constrained and the solver will return quickly from these queries.*
Another extremal case is when the solver returns satisfying assignments but the set KnownValues is being filled slowly, which means that the different satisfying assignments returned by the solver are very similar even though the queries are different. In such case the dependencies between variables must be loose and again, the solver is likely to find the solution quickly. In summary, these two intuitive observations suggest that if the solver is called many times, then the individual calls will be fast.

- *The number of calls to the SAT solver decreases if testing satisfiability of a variable reuses information obtained from previous tests.*

4.4 Modifying the Solver

Küchlin and Kaiser study how to compute the set of bound variables using a SAT solver in a non-interactive setting [113]. Their algorithm is based on similar principles as the algorithm Test-Vars, however, the state of the configuration process is not maintained. The optimization they use is based on modifying how the solver looks for solutions. This section shows how to implement this optimization in our setting.

Recall that the set *KnownValues* stores the literals that have occurred in the satisfying assignments found in the previous iterations of the loop in the function Test-Vars. And, if a literal is already in this set, it is not necessary to test if it is satisfiable in the current state formula because we know that it is. Hence, it is desirable to maximize the number of literals in the satisfying assignments that are not in the set *KnownValues*, in order to speed up its growth.

Example 16. *Let the state formula be equivalent to* True*, then it is possible to determine all the bound variables in two calls to the solver. In the first call the solver returns all negative literals and in the second call it returns all positive literals. Hence, the procedure* Test-SAT *never calls the solver again since all the literals are in the set KnownValues.*

In order to discover satisfying assignments that differ from the set of

KnownValues as much as possible, we modify how the solver searches for them. Recall that the solver is a backtracking algorithm systematically traversing the search space by making decisions (see Section 2.3). Therefore, to achieve our goal, we modify the solver to prefer decisions that take it to the parts of the search space that differ in from the literals known to be satisfiable.

We are assuming that the underlying solver is based on the principles outlined in Section 2.3. The solver is traversing the search space of Boolean assignments by backtracking while using unit propagation to avoid parts of the search space without a solution. If unit propagation has been carried out but there are still some variables that were not assigned a value, then the backtracking algorithm needs to choose a variable and a value for it in order to continue with the search (if the search is not successful, the value of that variable will be flipped).

It is common to represent a variable and its value as a literal (v corresponds to the value TRUE $\neg v$ corresponds to the value FALSE). We maintain a list of literals *Order*. The order of the literals in this list determines how they are chosen in backtracking. Hence, the part of backtracking that decides what is the next part of the search space to inspect is modified as shown by the following pseudo-code.

```
SELECT() : Literal
1  foreach k in Order
2        do if k has not been assigned a value
                 by the previous steps of backtracking
3              then return k
```

To initialize and update the list *Order*, we modify the routines INIT and TEST-SAT (see Chapter 4). When the configurator is initialized, the procedure INIT fills the list *Order* with all possible literals in some random order (for each considered variable two literals are added). The list is modified in the function TEST-SAT in such way that literals in *KnownValues* are move towards the end of the list. This is shown by the following pseudo-code.

TEST-SAT(ψ: *CNF*, l: *Literal*) : *Boolean*

```
1  if l ∈ KnownValues then return TRUE
2  if l ∈ DisabledValues then return FALSE
3  L ← IsSAT(ψ ∧ l ∧ ⋀_{k∈DisabledValues} k̄)
4  if L ≠ null
5     then foreach k ∈ L
6                do if k ∉ KnownValues move k to the end in the list Order
7          KnownValues ← KnownValues ∪ L
8     else  DisabledValues ← DisabledValues ∪{l}
9  return L ≠ null
```

The difference between this version and the last one is in the lines 5–6. Whenever a literal k is added to the set *KnownValues*, it is moved to the end in the order list. Hence, when searching for solutions, the backtracking algorithm will look for solutions containing the literal k, only if other solutions were not found.

Example 17. *Let* $\phi \stackrel{\text{def}}{=} \neg x \vee \neg y \vee z$. *We can see that all variables can be either* TRUE *or* FALSE. *Now, let us see how this is derived using the proposed algorithm. Let Order be initialized as* $x, y, z, \neg x, \neg y, \neg z$. *First the procedure* TEST-VARS *invokes* TEST-SAT *on the literal* x. *Since both KnownValues and DisabledValues are empty, the function invokes the SAT solver on* $\phi \wedge x$. *Within the solver, the variable* x *is given the value* TRUE *by unit propagation and the solver invokes the function* SELECT *to obtain the literal on which to backtrack next. Since the first literal in Order that has not been assigned a value yet is* y, *it is returned. Now, unit propagation assigns the value* TRUE *to the variable* z. *Hence, we have the model* $L = \{x, y, z\}$. *These literals are added to KnownValues and the list Order is updated to* $\neg x, \neg y, \neg z, x, y, z$.

Next, the procedure TEST-VARS *is invoked on the literal* $\neg x$. *Since the literal is neither in KnownValues or DisabledValues, the solver is invoked on* $\phi \wedge \neg x$. *Within the solver, the variable* x *is given the value* FALSE *by unit propagation and the function* SELECT *returns the literal* $\neg y$. *Now both* x *and* y *have the value* FALSE *and therefore unit propagation does not do anything. Thus, the solver invokes* SELECT *again, and,* $\neg z$ *is returned. The solver returns the model* $L = \{\neg x, \neg y, \neg z\}$. *These literals are added to KnownValues, which now contains all possible literals:* $\{x, y, z, \neg x, \neg y, \neg z\}$. *Therefore, any subsequent call to* TEST-SAT *will immediately return* TRUE *(line 1) and the solver does not need to be invoked.*

4.5 Producing Explanations

There are different types of explanations that a configurator can provide. The most basic type is *the set of user decisions* that necessitated a certain variable to be locked. This can be made more informative by also including the explanation *parts of the initial formula* responsible for the lock. Using a SAT solver enables even more than these two types: the explanation is a *tree of resolutions* (see Section 2.1.3) with the leaves forming a subset of the user decisions and parts of the initial formula. Note that the tree contains both the decisions and parts of the initial formula that are responsible for a certain variable to be locked, but also, logical dependencies between them. Hence, we treat explanations as trees of resolutions. Note that the two, less informative, types of explanations can easily be obtained from such tree.

As the main objective of this dissertation is to develop configurators for feature models, it is assumed that the initial formula is a conjunct of formulas corresponding to the individual modeling primitives (see Section 2.4.3), which have been further translated into sets of clauses. Therefore, once the tool identifies the clauses in the explanation, it can track them back to the primitives and display those to the user; this translation is discussed in greater detail in Chapter 7. In this chapter, explanations are constructed from user decisions and clauses coming from the initial formula.

Recall that a tree of resolutions is a tree representing a repeated application of the resolution rule and that the root is a consequence of the leaves of that tree (Section 2.1.3). Hence, explanations are defined as resolution trees.

Definition 6. *Let ϕ_0 be a CNF formula and D be a set of literals corresponding to user decisions such that $\phi_0 \wedge \bigwedge_{d \in D} d \models l$ for some literal l. An* explanation *for $\phi_0 \wedge \bigwedge_{d \in D} d \models l$ is a tree of resolutions with the root being l and each leaf being a clause from ϕ or a literal from D.*

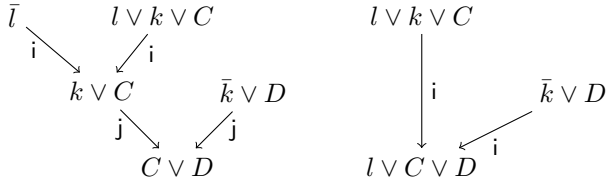
- *An explanation is a resolution tree certifying that a certain variable must be locked. The leaves of this tree are user decisions and parts of the initial formula.*

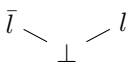
4.5.1 From Refutation to Explanation

An explanation for l can be requested only if $\phi \wedge \bar{l}$ has been shown to be unsatisfiable, where ϕ is the current state formula (Section 4.1). If that is the case, the configurator can send a request for a refutation of $\phi \wedge \bar{l}$ by sending the message Refute to the SAT solver. The refutation is a resolution tree where each of the leaves is a clause from the state formula ϕ or the unit clause $\bar{l}$; the tree's root is the empty clause $\bot$.

We show that a tree of resolutions with the root l, which is required by Definition 6, can always be obtained from such refutation tree. This construction hinges on *resolution skipping*, which is introduced by the following definition.

Definition 7 (Resolution Skipping). *Let T be a resolution tree, then T' is obtained from T by* resolution skipping *of l by applying the two following rules. (1) For any clause of the form $l \vee k \vee C$ which is first resolved with $\bar{l}$ and then with some clause $\bar{k} \vee D$, perform only the resolution with $\bar{k} \vee D$ (where C and D are some disjunctions of literals). This transformation is depicted below.*



(2) If the literal $\bar{l}$ is resolved with l in T, make l the root of T', i.e., the tree  *is transformed into l.*

Claim 4. *Let ϕ be a satisfiable CNF formula and l a literal for which $\phi \wedge \bar{l}$ is unsatisfiable and let T be a resolution tree showing the unsatisfiability, then the tree T' obtained by resolution skipping of $\bar{l}$ is a tree of resolutions that has l as the root.*

Proof. Since $\phi \wedge \bar{l}$ is unsatisfiable, there is a resolution tree T using clauses from ϕ and the clause $\bar{l}$. Since ϕ alone is satisfiable, T must contain the literal $\bar{l}$.

After any of the rules (1) and (2) of skipping is applied (Definition 7), the tree T' contains additionally the literal l in the place where the rule was applied. Since T' does not contain any resolutions over the literal $\bar{l}$, the literal l cannot be removed by any of the further resolutions. Since there is a path from $\bar{l}$ to $\bot$ in T, the tree T' contains l instead of $\bot$ (recall that $\bot$ is the empty clause). □

Figure 4.7 shows pseudo-code for realizing resolution skipping. The algorithm applies the skipping recursively and the results of the recursive call are resolved. If one of the children of the given node is $\bar{l}$, the other child is returned, which effectively means that the resolution over $\bar{l}$ is not performed.

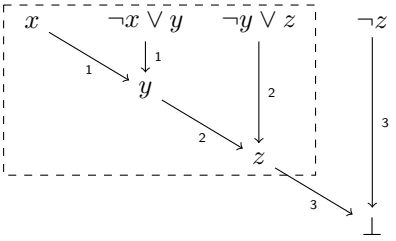
> ***Remark 4.14.*** *Since there may be sharing in the tree, the implementation of the algorithm is memoized. When a subtree does not contain the literal $\bar{l}$, the skipping leaves the tree as it is and therefore in such case the resolution at line 6 does not need to be replayed.*

In some cases, the tree of resolutions obtained by resolution skipping is already contained in the refutation tree as shown in Figure 4.8. This happens

▷ **ensures** if *tree* contains a resolution over $\bar{l}$,
then $result.resolvent = result.resolvent \vee l$
▷ **ensures** if *tree* does not contain a resolution over $\bar{l}$,
then $result = tree$
SKIP($tree : Tree; l : Literal$) : *Tree*

```
1  if tree is a leaf then return tree
2  child_l ← SKIP(tree.left, l)
3  child_r ← SKIP(tree.right, l)
4  if child_r . resolvent = l̄ then return child_l
5  if child_l . resolvent = l̄ then return child_r
6  return RESOLVE(child_r, child_l)
```

Figure 4.7: Realizing resolution skipping



A refutation of $\phi \wedge \neg z$, which shows that $\phi \models z$. The dashed rectangle marks the part of the tree of resolutions obtained by resolution skipping of $\bar{z}$.

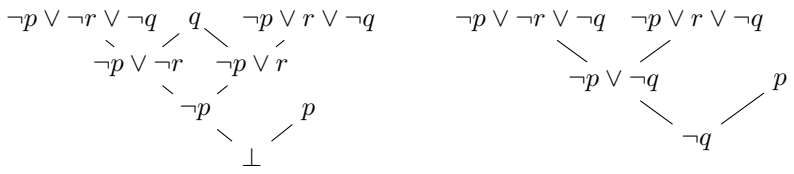
Figure 4.8: A resolution tree used to explain a locked variable

whenever the resolution over $\bar{l}$ is the last one in the resolution tree, i.e., it is the resolution yielding the resolvent $\bot$. The following example shows that resolution skipping may lead to more complex transformations.

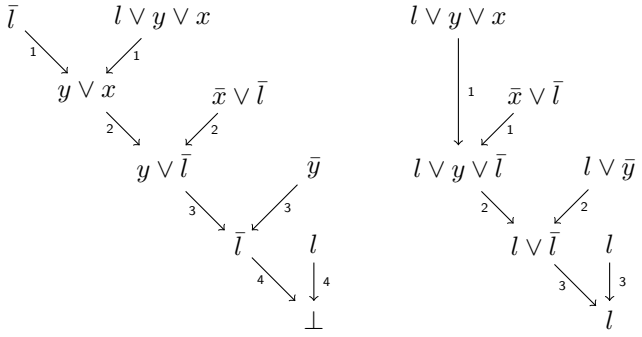
Example 18. *Let us consider the following state formula ϕ*

$$\begin{array}{l} \neg p \vee \neg r \vee \neg q \\ \neg p \vee r \vee \neg q \\ \quad p \end{array}$$

It holds that $\phi \models \neg q$. The tree on the left below refutes $\phi \wedge q$ and the tree on the right is the corresponding explanation obtained by resolution skipping.



Remark 4.15. *If T' is obtained from T by resolution skipping, it may be that T' contains clauses containing both the literal l and $\bar{l}$, which semantically corresponds to* True. *For illustration look at the trees below; the one on the left is before resolution skipping and the one on the right after:*



Logically, the inference is correct but there are some redundant steps. The proof obtained from the solver derives $\bar{l}$ from $\bar{l}$ in a complicated fashion. The tree obtained by skipping infers l from True *and l, hence that part of the subtree can be dropped—in this case we obtain a single-node tree as l was part of the formula. In general we drop the trees that derive* True *as well all the resolution steps that become invalid because of that.*

- *An explanation in the form of a tree of resolutions can be obtained from a refutation tree by resolution skipping.*

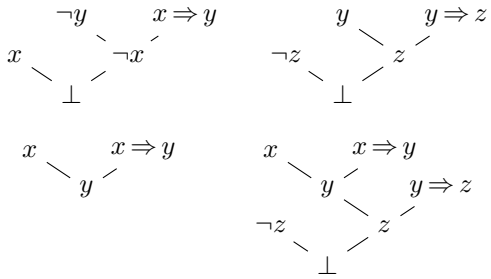
4.5.2 Reconstructing Resolution Trees and The Catalyst Optimization

If the catalyst optimization is used, the algorithm Test-Vars conjoins to the state formula previously inferred literals (see line 4 in Figure 4.5). Therefore, the proof returned by the solver may rely on these literals—meaning that they appear in the leaves. This poses a problem for producing explanations as these literals might not be part of the state formula, which comprises the initial formula and user decisions. Recall that we require the explanation to have in the leaves only clauses from the state formula (see Definition 6).

Hence, we need to reconstruct the resolution tree so that the catalyst literals are no longer in the leaves. This is illustrated by the following example.

Example 19. *Let $\phi \stackrel{\text{def}}{=} (x \Rightarrow y) \wedge (y \Rightarrow z) \wedge x$ where x is the only user decision. The procedure* Test-Vars *calls the solver with the query $\phi \wedge \neg y$, which is unsatisfiable and therefore $\phi \models y$. In the next iteration the procedure calls*

the solver with $\phi \wedge y \wedge \neg z$ to determine whether z can be FALSE*, to which the solver responds negatively. As the solver has the previously inferred literal y at its disposal, the proof it returns relies only on y, $y \Rightarrow z$, and $\neg z$. Since y is inferred and not given by the user, the configurator must add the explanation for y. The resolution trees obtained from the refutations are depicted below in the first row. The tree on the left shows that $\phi \wedge \neg y$ is unsatisfiable and the tree on the right shows that $\phi \wedge y \wedge \neg z$ is unsatisfiable.*



At this point the tree refuting $\phi \wedge y \wedge \neg z$ does not satisfy the requirements on the leaves stated earlier (see Definition 6) as the leaf y is neither a user decision nor part of the initial formula. It is necessary to merge the two trees in order to justify y in the explanation of z.

To construct the tree required in the example above, we perform the two following steps. (1) We perform resolution skipping for $\phi \wedge \neg y$ so that y becomes the root (instead of $\bot$). (2) We attach this transformed tree to the resolution tree for $\phi \wedge y \wedge \neg z$ in order to justify y. The tree obtained in step (1) is depicted the second row above on the left and how it is attached in step (2) is on the right.

```
▷ requires φ ∧ l_1 ∧ ··· ∧ l_k ∧ l̄ is unsatisfiable
▷ requires φ ∧ l_1 ∧ ··· ∧ l̄_i is unsatisfiable for i ∈ 1..k
RECONSTRUCT(l_1, ..., l_k : Literal; l : Literal) : Tree
1  T ← REFUTE(φ ∧ l_1 ∧ ··· ∧ l_k ∧ l̄)
2  L ← {l_i | i ∈ 1..k and l_i is a leaf in T}
3  foreach l_i ∈ L
4        do T_i ← RECONSTRUCT(l_1, ..., l_{i−1}; l_i)
5           T'_i ← SKIP(T_i, l_i)
6           Replace the leaf l_i in T with T'_i
7  return T
```

Figure 4.9: Justifying previous inferences in an explanation

To conclude, let us show how a refutation tree for $\phi \wedge \bar{l}$ is reconstructed in general. The situation we consider is that $\phi \wedge l_1 \wedge \cdots \wedge l_k \wedge \bar{l}$ is unsatisfiable, where

$l_1, \ldots, l_k$ are previously inferred catalyst literals in this order. The pseudo-code in Figure 4.9 shows how to compute the resolution tree for $\bar{l}$ that has none of the $l_1, \ldots, l_k$ in the leaves. The function RECONSTRUCT first asks the solver for a resolution tree, which might contain the undesired leaves (line 1). For each such leaf it constructs a refutation with a recursive call (line 4) and performs resolution skipping (line 5) so it contains the literal l_i as the root by calling the function SKIP (see Figure 4.7). The transformed tree is then used as a justification of the leaf (line 6).

> ***Remark 4.16.*** *Since any literal l_i can appear in the explanation of $l_{i+1}..l_k$ multiple times, the implementation of* RECONSTRUCT *should be memoized.*

- *If the catalyst optimization is used, the catalyst literals in the leaves of the resolution tree need to be replaced with their explanations.*

4.5.3 Notes on Explanation Size and Complexity

The explanations are constructed using resolution trees obtained from a SAT solver. One issue with such a tree is that it might *not* be minimal. That is, minimal in its size or even in the sense that removing certain clauses or user-decisions still yields an unsatisfiable formula. Naturally, for the user-friendliness sake, it is desirable for the explanation to be small. There is a significant amount of research on obtaining small proofs and the reader is referred to the chapter on related work for further references (Section 10.1.2). Any of these techniques can be applied to make the explanations smaller. The implementation pertaining to this dissertation uses for proof minimization the QuickXplain algorithm [111], which is built in the solver SAT4J, and the tool MUSER [140] (see Section 7.1 for more details).

Another related question is whether it is not possible to provide explanations with a simpler mechanism than resolution trees. Unit propagation provides ways how to implement a non-backtrackfree configurator. In such configurator the explanation mechanism can be implemented in an elegant way [15] (see also Section 10.1).

This might suggest that explanations could be provided by reusing this technique. However, since unit propagation in general does not infer all bound literals, such approach would require adding to the formula some information obtained from the SAT solver. One possibility is to add the bound literals obtained by using a SAT solver. The issue with this approach is that an explanation for a certain bound literal obtained from unit propagation will comprise that literal and nothing else. Hence, we would be getting explanations that are not useful.

A more general issue relates to a result by Cook and Reckhow according to which proofs are super-polynomial in size of the original formula unless *co-NP≠NP* [42]. Since unit propagation always performs a polynomial number of steps, it means that for unit propagation to be able to infer any possible bound literal, a super-polynomial information must be added.

- *From the theoretical perspective, proofs, and therefore explanations, must be in general super-polynomially large. However, proof-minimization techniques can be applied to mitigate this problem.*

4.6 Summary

This chapter shows how to utilize a SAT solver to implement a complete and backtrack-free configurator. The state of the configuration is maintained in the *state formula*, which is a conjunct of the semantics of the instance being configured and current user decisions. The algorithm hinges on propositions established in Chapter 3; namely, that a literal is bound if and only if conjoining the complementary literal with the state formula yields an unsatisfiable formula. A simple algorithm is constructed from this proposition that invokes a SAT solver on each considered literal (Section 4.2). This algorithm is made more efficient by availing of the fact that a SAT solver returns a satisfying assignment for satisfiable queries (Section 4.3).

Section 4.4 shows how to integrate in the algorithm an optimization that by Küchlin and Kaiser used in the context of non-interactive computation of bound literals. This optimization modifies the order in which the underlying SAT solver traverses the search space of variable assignments and it is aimed at reducing the number of calls to the solver.

Section 4.5 how to provide explanations to the user why a certain variable was locked in a certain value. Explanations are defined as trees of resolutions that are rooted in the literal to be explained. The leafs of such tree comprise the clauses from the state formula responsible for the lock and the internal nodes are chains of reasoning for the locked literal from these leaves. It is shown that such tree can always be obtained from a resolution tree returned by the underlying solver (Figure 4.7). Since not all SAT solvers provide resolution trees, an alternate solution for obtaining them is described in the chapter concerned with implementation (Section 7.1).

In summary, this chapter makes the two following contributions: algorithms that enable us to construct a backtrack-free and complete configurator using a SAT solver and algorithms that enable us to construct explanations for why a certain decision is disabled from resolution trees.

Chapter 5

Bi-Implied Set Optimization

This chapter describes a syntactic-based optimization aimed at reducing the number of satisfiability tests in the procedure TEST-VARS (Figure 4.3). This optimization is a variation of another optimization known in the feature modeling community as *atomic sets* [207, 180]. Atomic sets are equivalence classes of features formed by the rule that a mandatory subfeature is in the same atomic set as its parent. The features in an atomic set always all appear in a permissible configuration or not at all, since a mandatory feature must be selected whenever its parent is. Thus, it is sufficient to consider only one of the features from each atomic set when analyzing the model, effectively collapsing the atomic sets. According to experimental research of Segura, collapsing atomic sets significantly reduces computation time of analyses [180].

We observe that the FODA semantics (Section 2.4.3) prescribes that $v_c \Leftrightarrow v_f$ if c is a mandatory child of f where v_c, v_f are the corresponding variables. Hence, the variables corresponding to the features in an atomic set all have the same value in any satisfying assignment of the formula representing the semantics of the feature model. This motivates us to generalize atomic sets into the context of CNF by focusing on *equivalent literals*, which are established by the following definition.

Definition 8 (Equivalent literals). *Given a formula ϕ, we say that the literals l and l' are equivalent in ϕ if and only if under every satisfying assignment of ϕ the literals l and l' evaluate to the same value (both are either* TRUE *or both* FALSE*). In mathematical vernacular $\phi \models l \Leftrightarrow l'$.*

Remark 5.17. *If a formula is unsatisfiable, i.e., has no satisfying assignments, all pairs of literals are equivalent. However, as we are interested in configuring satisfiable formulas, this is not practically important.*

Recall that the configurator described in the previous chapter determines which literals are bound in the current state formula by using satisfiability tests (see TEST-VARS in Figure 4.3). The optimization developed in this section collapses equivalent literals in order to decrease the number of these satisfiability tests. The following observation shows that this is possible.

Observation 5. *Let ϕ be a formula and l, l' be literals equivalent in ϕ, and let ϕ' be ϕ with the literal l' substituted by l. Then the formulas ϕ and ϕ' are equisatisfiable.*

Proof. If there is satisfying assignment for ϕ, then this assignment is also satisfying for ϕ' because l and l' evaluate to the same value under this assignment.

If there is a satisfying assignment for ϕ', this can be extended into a satisfying assignment of ϕ by assigning such value to the variable in l' that l and l' evaluate to the same value. □

The observation above tells us that to compute the satisfiability of $\phi \wedge l_i$ for some set of equivalent literals $l_1, \ldots, l_n$, it is sufficient to consider only one of the literals since all the other literals yield the same result. Clearly, it is beneficial to identify sets of equivalent literals that are as large as possible. In general, however, identifying equivalent literals is expensive complexity-wise as shown by the following claim.

Claim 5. *Determining whether two literals are equivalent in a given formula is co-NP hard.*

Proof. Let ϕ be a formula. We show that the test for whether ϕ is a tautology can be converted to a literal equivalence test. Let x and y be variables not appearing in ϕ and let $\phi' \stackrel{\text{def}}{=} (\phi \Leftrightarrow x) \wedge y$. If ϕ is a tautology, then any variable assignment of the variables in ϕ can be extended to a satisfying assignment of ϕ' where both x and y are TRUE, and, these are the only satisfying assignments of ϕ'. Note that if x and y appeared in ϕ, it would not be guaranteed that any satisfying assignment of ϕ can be extended to ϕ', e.g., for $\phi \equiv \neg x \vee x$ or $\phi \equiv \neg y \vee y$.

Conversely, if x is TRUE in every model of ϕ' then ϕ must be a tautology. Consequently, ϕ is a tautology iff x is TRUE in all models of ϕ'. And, since y is TRUE in all models of ϕ', the literals x and y are equivalent iff ϕ is a tautology.

In summary, deciding the equivalence of x and y solves a co-NP-complete problem and the reduction took polynomial time and thus the problem of literal equivalence is co-NP hard. □

Discouraged by the above claim, we look only for *some* pairs of equivalent literals—for which it is easier to tell that they are equivalent. For inspiration we look again at atomic sets. The semantics for mandatory subfeature yields the bi-implication $v_f \Leftrightarrow v_c$, which is equivalent to $(v_f \Rightarrow v_c) \wedge (v_c \Rightarrow v_f)$. Another way of seeing this last formula is that these are two literals implying each other.

Hence, instead of just exploiting the mandatory relation, we consider literals and implications between them. If literals imply each other, then they are equivalent. Moreover, literals in a cycle of implications are equivalent due to transitivity of implication.

To obtain implications from a CNF formula, we observe that every two-literal clause corresponds to two implications.

Observation 6. *Any clause comprising two literals corresponds to two implications, which are contrapositives of each other.*

Example 20. *The formula $(x \vee y) \wedge \neg z$ contains two clauses. The clause $x \vee y$ corresponds to the implications $\neg x \Rightarrow y$ and $\neg y \Rightarrow x$.*

The set of implications obtained from two-literal clauses give rise to a graph on the literals. This graph is defined as follows.

Definition 9 (Implication graph). *The implication graph [7] of a CNF formula ψ is obtained by introducing the vertices x and $\neg x$ for each variable x appearing in ψ and adding the edges $\langle \bar{l}_1, l_2 \rangle$ and $\langle \bar{l}_2, l \rangle$ for all clauses of the form $l_1 \vee l_2$ in ψ.*

> ***Remark 5.18.*** *Recall that $\bar{l}$ denotes the complementary ("negated") literal of l (see Section 2.1.3). The implication graph construction ignores clauses with less or more than two literals.*

In the following discussion we utilize the graph theory concept strongly connected components. Recall that a ***s**trongly **c**onnected **c**omponent* (*SCC*) in a directed graph is a maximal subgraph in which there is a path between each two nodes of that subgraph.

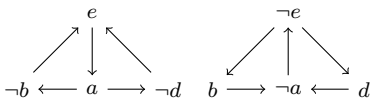
Definition 10 (Bi-Implied Sets). *In the context of a formula ϕ,* bi-implied literals *are literals that appear in the same SCC of the implication graph of ϕ. We call the SCCs of an implication graph* **bi-implied sets** *(BISs)*

The following examples illustrate the defined terms. Note that the examples in this section treat CNF formulas as a set of implicitly conjoined clauses in

order to avoid cluttering the text.

Example 21. *The following set of clauses determines an implication graph comprising two BISs depicted on the right.*

$$\neg a \vee \neg b \qquad e \vee d$$
$$\neg a \vee \neg d \qquad \neg e \vee a$$
$$e \vee b$$

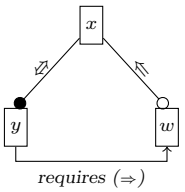


Example 22. *Consider the formula* $q \wedge x \wedge x \Rightarrow (y \Rightarrow z) \wedge x \Rightarrow (y \Rightarrow z)$ *corresponding to the set of clauses* $\{q, x, \neg x \vee \neg y \vee z, \neg x \vee \neg z \vee y\}$*. The literals* q *and* x *are equivalent. In fact, they are both bound in the formula—they are always* TRUE *due to the unit clauses* q *and* x*. The literals* y *and* z *are equivalent because they imply each other under the condition that* x *is* TRUE*, which is always fulfilled. However, none of the equivalent literals are in the same BIS.*

The following observation summarizes the message of the examples above.

Observation 7. *The literals in the same BIS of the formula* ϕ *are equivalent in* ϕ*. However, there may be literals that are equivalent and are not in the same BIS.*

Example 23. *Consider the fragment of a feature diagram depicted below (adorned with the connectives corresponding to the semantics). There are two atomic sets:* $\{x, y\}$ *and* $\{w\}$*. However, we can detect that* x*,* y*, and* w *are equivalent by looking at the implication graph (they are all in the same BIS).*



Observation 8. *The literals corresponding to variables in an atomic set always belong to the same BIS. However, there may be other literals in such BIS.*

Note that the two BISs in Example 21 are *dual* to one another: edges are reversed and literals complemented. That is not a coincidence, the reason for that is that each two-literal clause corresponds to *two* dual implications (contrapositives).

Observation 9. *For any satisfiable formula and for each variable* v *there is a BIS containing the literal* v *and a dual BIS containing the literal* $\neg v$*, where a dual BIS is obtained by reversing all the edges and replacing the literals with their complements.*

Remark 5.19. *If for a formula ϕ the literals v and $\neg v$ are in the same BIS, ϕ must be unsatisfiable since v and $\neg v$ have the same value in all models of ϕ, which is possible only if ϕ has no models. See also Remark 5.17.*

Observation 10. *Computing BISs can be done in linear time of the size of the formula since collecting the implications is a simple traversal of the formula and computing the SCCs of the implication graph can be done in linear time [190].*

- *Identifying Bi-Implied Sets (BIS) is a computationally cheap way of identifying some, but not all, equivalent literals.*

5.1 Performing BIS-Optimization

The BIS-optimization picks one literal from each BIS to represent the rest. This literal is called the *representative* and in the rest of this section $l \mapsto r$ denotes that the literal l *is represented* by the literal r. Since the literals in any BIS are equivalent, replacing a literal with its representative yields an equisatisfiable formula.

Example 24. *Let $\phi \stackrel{\text{def}}{=} \{x \Rightarrow y, y \Rightarrow z, z \Rightarrow x, x \vee y \vee z\}$. Let x represent the literals x, y, and z (which are in the same BIS of ϕ). In the context of ϕ, instead of $x \vee y \vee z$, the BIS optimization performs computations on x.*

As the goal is to minimize the number or variables, the optimization chooses the representatives in such a way that:

1. All literals in the same BIS have the same representative.
2. Complementary literals are used to represent dual BISs.

Hence, for Example 21, the literal a will represent the BIS on the left if and only if the literal $\neg a$ represents the BIS on the right.

Remark 5.20. *Collapsing literals sometimes causes clauses to collapse as well. For instance, substituting y for x and $\neg y$ for $\neg x$ in the CNF formula $\{x \vee y, \neg x \vee y, y\}$ yields the CNF formula $\{y\}$.*

The following example in following illustrates how the BIS-optimization is used in a configurator.

Example 25. *Consider the CNF formula $\phi \stackrel{\text{def}}{=} \{f_1 \Rightarrow f_2, f_2 \Rightarrow f_1, \neg f_1 \vee \neg f_3, f_3\}$. As f_1 and f_2 imply one another, they fall into the same BIS. We choose f_1 to represent f_2, i.e., $f_2 \mapsto f_1$, and perform the pertaining substitution, which yields the formula $\{f_1 \Rightarrow f_1, f_1 \Rightarrow f_1, \neg f_1 \vee \neg f_3, f_3\}$. The first two implications do not affect the set of bound literals since both of the clauses are trivially* TRUE.

Hence, these implications are automatically discarded yielding the optimized formula: $\phi' \stackrel{\text{def}}{=} \{\neg f_1 \vee \neg f_3, f_3\}$.

Now we want to determine the satisfiability of $\phi \wedge f_2$. *From the mapping* $f_2 \mapsto f_1$ *we know that we need to ask the solver for the satisfiability of* $\phi' \wedge f_1$. *The SAT solver replies that* $\phi' \wedge f_1$ *is unsatisfiable (*$\neg f_1$ *is bound in* ϕ'*). Hence,* $\phi \wedge f_2$ *is unsatisfiable as well due to Observation 5 (*$\neg f_2$ *is bound in* ϕ*).*

What if we want to know why $\neg f_2$ *is bound? Again, relying on the mapping, we ask why* $\neg f_1$ *is bound* ϕ'. *This can be explained using the clauses* $\{\neg f_1 \vee \neg f_3, f_3\}$. *To relate the explanation to* $\neg f_2$, *we also add the implication* $f_2 \Rightarrow f_1$.

When initialized with the initial formula, the BIS-optimization computes the BISs and records the choice of representatives. Then, whenever the satisfiability of ψ needs to be determined, the satisfiability of ψ' is computed instead, where ψ' is ψ with all literals substituted with their representatives (see also Observation 5). This is captured by the function Test-SAT-BIS (Figure 5.1) which replaces the function Test-SAT in Test-Vars (Figure 4.3). The function first performs the substitution of literals and then uses the original function Test-SAT. Recall that Test-SAT never invokes the SAT solver more than once for the given literal (Observation 2) and that the solver is not called more than $n + 1$ times, where n is the number of variables (Claim 3). Hence, the BIS-optimization does not call the solver more than $n + 1$ where n is the number of variables in the optimized formula.

```
Test-SAT-BIS(ψ: Formula, l: Literal) : Boolean
    ψ' ← substitute each literal with its representative in ψ
    r ← representative of l
    return Test-SAT(ψ' ∧ r)
```

Figure 5.1: Satisfiability testing with BIS-optimization

Remark 5.21. *It is possible to integrate the BIS-optimization into a SAT solver (see Section 10.2 for further references). However, if that is the case, the effect is not the same as applying the optimization upon initialization of the configurator. In the way we use the optimization, not only the size of the formula is reduced, but also, the solver is invoked fewer times—once we know whether a literal is bound or not, we know that the same holds for all the literals in the same BIS. Hence, if the solver implements the BIS-optimization internally, it is not as helpful. Firstly, in such case our algorithm calls the solver the same number of times as if the optimization is not implemented. Secondly, the solver needs to perform the optimization each time it is invoked, while in our case it is performed just once.*

As computing explanations in the BIS-optimization is more subtle, this is discussed separately in the following section.

- *The BIS-optimization collapses literals occurring in the same Strongly Connected Component of the implication graph.*

5.2 Computing Explanations

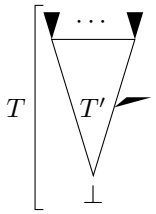
To explain why a certain variable is bound, the configurator uses a resolution tree (see Section 4.5). If the BIS optimization is used, the solver operates on the optimized formula and knows nothing about the original, non-optimized version. In particular, any resolution tree obtained from the solver is on the optimized version. Therefore, in order to provide explanations on the non-optimized formula, we need to *reconstruct* the resolution tree for the non-optimized formula from the resolution tree for the optimized version.

$$\begin{array}{ccc} \phi \wedge \bar{l} & \xrightarrow{\text{optimize}} & \phi' \wedge \bar{r} \\ \text{refute}\uparrow & & \uparrow\text{refute} \\ T & \xleftarrow[\text{reconstruct}]{} & T' \end{array}$$

Figure 5.2: Depiction of reconstruction of a proof in the context of the BIS-optimization where T and T' are resolution trees refuting the respective formula

Let us look more closely at what happens (see Figure 5.2). Let ϕ be the non-optimized formula and l be a literal such that $\phi \wedge \bar{l}$ is unsatisfiable (equivalently $\phi \models l$). If ϕ' is the optimized version of ϕ and r is the representative of l, then $\phi' \wedge \bar{r}$ is unsatisfiable as well (Observation 5). Therefore we can ask the solver to give us a resolution tree for $\phi' \wedge \bar{r}$ and use it to build a tree for $\phi \wedge \bar{l}$.

The basic idea for the tree reconstruction is that we keep the refutation of the

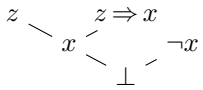


A schematic depiction of the reconstruction of a tree from the optimized formula. The resolution tree T' comes from reasoning on the optimized formula. The smaller, filled trees come from the BISs.

Figure 5.3: Proof reconstruction

optimized formula and enhance it with implications taken from the implication graph as these relate the non-optimized to the optimized. What makes the problem interesting is that it is not necessary to add all the implications as illustrated by the following example.

Example 26. *Let $\phi \stackrel{\text{def}}{=} \{\neg x, x \Rightarrow y, y \Rightarrow z, z \Rightarrow x\}$ and let x represent the literals y and z, then the optimized version ϕ' is $\{\neg x\}$. Since $\phi' \wedge x$ is unsatisfiable and x is the representative of z, then $\phi \wedge z$ is unsatisfiable. The refutation of $\phi' \wedge x$ is*  *. To obtain a refutation of $\phi \wedge z$, it is sufficient to resolve z with the implication $z \Rightarrow x$, to derive that the representative x holds while the rest of the steps remain the same. This yields the following tree.*



In general, the tree reconstruction takes the tree refuting the optimized form and prepends it with chains of implications for representatives appearing in the optimized refutation (see Figure 5.3).

The following text explains the proof reconstruction in a top–down style. The top-level function is BIS-Proof which produces a resolution tree for a given formula and literal.

▷ **requires** $\phi \wedge l$ is unsatisfiable
BIS-Proof(ϕ : *CNF*; l : *Literal*) : *ResolutionTree*
1 let ϕ' be the optimized version of ϕ
2 let r be the representative of l
3 let T' be resolution tree for $\phi' \wedge r$
4 $T \leftarrow T'$
5 for each clause c that is a leaf in T' call SwapAll(c)
6 **return** T

The parameters of the function BIS-Proof are a CNF formula and a literal such that the conjunction of the literal and the formula is unsatisfiable. The function computes a resolution tree on the optimized version (line 3) and then it

calls the function SwapAll to prepend the leaves of that tree with appropriate implications (line 5).

The function SwapAll used in BIS-Proof calls the function Swap multiple times in order to derive a clause comprising the representatives of the literals in the clause given to the function.

```
SwapAll(c : Clause) : Clause
    c' ← c
    foreach l in c
        do c' ← Swap(c', l)
    return c'
```

The function Swap called on each literal of a clause in SwapAll constructs a tree of resolutions that replaces a literal in the given clause with its representative. The pseudo-code assumes there is a function Resolve that performs a resolution step on two given clauses and records it in the resolution tree T that is being built by BIS-Proof (see above).

```
▷ requires l occurs in c
▷ returns {r} ∪ c ∖ {l}, where l ↦ r
Swap(c : Clause; l : Literal) : Clause
 1  let r be a representative of l
 2  let P be a sequence of implications forming
      some path from l to r in the implication graph
 3  c' ← c
 4  k ← l
 5  ▷ invariant: c' ≡ {k} ∪ c ∖ {l}
 6  ▷ invariant: the implications of P form a path from k to r
 7  while P ≠ ∅
 8        do let k̄_i ∨ k_{i+1} be the first element of P
 9           c' ← Resolve(c', k̄_i ∨ k_{i+1})
10           let k ← k_{i+1}
11           remove (k̄_i ∨ k_{i+1}) from P
12  return c'
```

Since the functions above hinge on the function Swap we conclude with a proposition proving correctness of Swap.

Proposition 4 (Literal Swap). *The function* Swap *returns* $\{l\} \cup c \setminus \{r\}$ *as expected.*

Proof. The invariants are established before the loop is entered because c' is equal to c, the literal k is equal to l, and P forms a path from l to the literal r.

If $P \equiv \bar{k}_i \vee k_{i+1}, \bar{k}_{i+1} \vee k_{i+2}, \ldots$ then the literal k_i must be in the clause c' according to the invariant. Resolving c' with $\bar{k}_i \vee k_{i+1}$ yields the clause $\{k_{i+1}\} \cup c' \setminus \{k_i\}$. The resolution preserves the invariant because now P forms a path from k_{i+1} to r and the literal k_{i+1} replaced k_i in c'.

Since the invariant on the loop of SWAP is correct and the loop terminates when P is empty, the clause c' must contain the literal r, which is the only literal by which it differs. □

- *To get the explanations for the original, non-optimized formula, some implications from the implication graph are added to the explanation of the optimized formula.*

Computing Explanations with a SAT Solver

An alternative way of computing the explanation is to use the SAT solver again on the unoptimized formula. This approach is less efficient and less predictable but easier to implement than the one described above.

If the optimized formula $\phi' \wedge r$ is unsatisfiable, then the original formula $\phi \wedge l$, where $l \mapsto r$, is also unsatisfiable (Observation 5). Therefore calling the solver on $\phi \wedge l$ will also yield a proof.

This approach is useful only when the explanations are needed significantly less often than the other SAT queries otherwise that would defeat the purpose of the optimization.

5.2.1 Choice of The Representative and Explanation Sizes

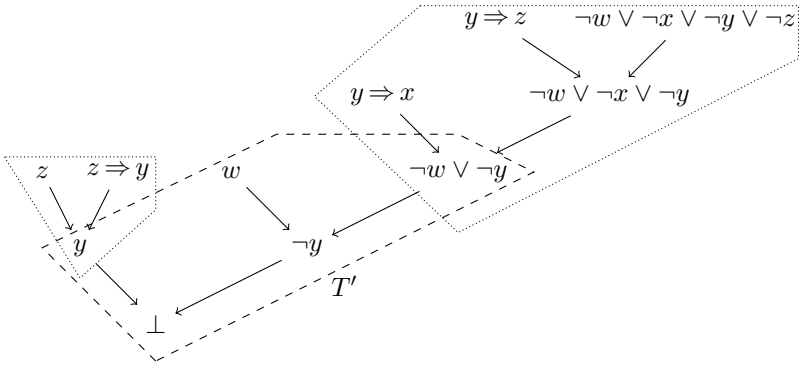
The BIS-optimization as described up to now, does not specify two important things: (1) How to choose a representative of a BIS. (2) How to choose a path from the literal to its representative in the proof reconstruction (SWAP). This section shows that both (1) and (2) affect the size of the explanations (resolution trees), and suggests a strategy for them. The following example shows how explanations vary depending on the choice of representative.

Example 27. *Let* $\phi \stackrel{\text{def}}{=} \{\neg w \vee \neg x \vee \neg y \vee \neg z, w\} \cup \{x \Rightarrow y, y \Rightarrow x, y \Rightarrow z, z \Rightarrow y\}$, *which tells us that at least one of* w, x, y, *or* z *must be* FALSE. *But also that* w *is always* TRUE *and that* x, y, *and* z *are equivalent. Therefore* $\phi \wedge l$ *is unsatisfiable for* l *being any of the* x, y, *and* z. *Let us see what happens when BIS-optimizations is applied. The BISs in the formula are the following.*

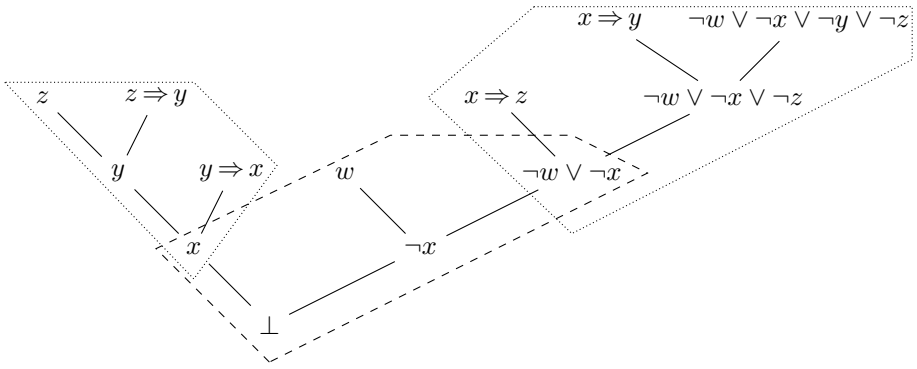
Let us assume that we want to show that z *must be* FALSE, *i.e., we need a resolution tree using* z *and* ϕ. *We investigate two scenarios: one when the*

representative for the literals $\{x, y, z\}$ is the literal y and the other when the representative is the literal x.

If y is chosen as the representative, ϕ is optimized into $\phi'_y \stackrel{\text{def}}{=} \{w, \neg w \vee \neg y\}$. The following is the resolution tree for $\phi \wedge z$ where the dashed region marks the resolution tree of $\phi'_y \wedge y$ and the dotted regions are the added trees that reconstruct the tree for $\phi \wedge z$, i.e., the dotted sub-trees are obtained by invoking SwapAll *on the clause in the root of the sub-tree.*



In contrast, if x is chosen as the representative, ϕ is optimized into $\phi'_x \stackrel{\text{def}}{=} \{w, \neg w \vee \neg x\}$. The following is also a resolution tree for $\phi \wedge z$ but reconstructed from a tree for $\phi'_x \wedge x$.



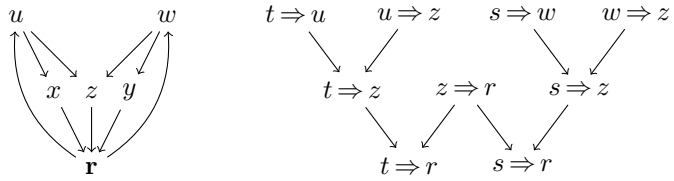
Note that the second tree requires one more resolution step to derive the representative compared to the first tree (the first tree has 11 nodes and the second 13). The reason is that to reach the representative x from z in the BIS $x \rightleftharpoons y \rightleftharpoons z$ requires two steps, whereas the representative y can be reached from z in one step. The following table summarizes how the different choices of representatives affect the size of the resolution tree. The rows correspond to different choices of representatives; the columns correspond to different satisfiability tests.

repr./test	x	y	z	*avg.*
x	9	11	13	11
y	11	9	11	$10.\bar{3}$
z	13	11	9	11

We can see that the representative y yields the lowest average over the queries x, y, and z. This result suggests that it is good to choose representatives that are in the "middle" of the BIS.

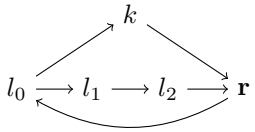
Apart from the choice of the representative, the size of the explanation is affected by the paths that are chosen in SWAP in order to get from a literal to its representative. This is illustrated by the following example.

Example 28. *Consider the two BIS depicted on the left below where r was chosen to be the representative (typeset in bold).*



The paths $u \to x \to r$ and $u \to z \to r$ are both shortest paths from u to r. If any of the two paths is chosen in the reconstruction, it takes two resolution steps to get from u to r. The same holds for the paths from w to r. However, if both u and w appear in the resolution tree, it is better to choose the paths going via z because the formula $z \Rightarrow r$, corresponding to the edge $z \to r$, is reused for both u and w therefore the overall size of the explanation is smaller. The resolutions depicted on the right illustrate how the implication $z \Rightarrow r$ can be reused within a tree using both w and u.

The BIS below illustrates that choosing a shortest path does not necessarily mean shorter explanations.



If all the literals l_0, l_1, l_2 appear in the resolution tree, it is better to choose the path $l_0 \to l_1 \to l_2 \to r$ to explain why r must hold if l_0 holds because the implications $l_1 \Rightarrow l_2$ and $l_2 \Rightarrow r$ have to be added to the explanation anyhow. If, however, the literals l_1 and l_2 do not need to be in the tree, then it is beneficial to choose the shorter path $l_0 \to k \to r$.

From the examples above we know that both the choice of the representatives and the choice of paths from literals to their representatives affect the size of explanations. The goal of this subsection is to find such strategies for these choices that they yield, or are expected to yield, small resolution trees on average. First let us focus on the problem of finding the paths with the representatives fixed. The following proposition makes the problem easier by revealing that it is sufficient to focus on each BIS separately.

Proposition 5. *Let l be a literal represented by r in the BIS S and let S' be a BIS different from S, then there is no path connecting l to r going through S'. In other words, a path from a l to r never leaves the BIS.*

Proof. (by contradiction) Let $l' \in S'$ such that the path from l to r goes through l'. Then there are paths from l to l' and from l' to r. Since r and l are in the same strongly connected component, there is also a path from r to l. Therefore there is a path from l' to r and a path from r to l' via l, which is a contradiction because that means that l' and r are in the same strongly connected component. □

For the reconstruction of a resolution tree we need paths of implications going from the literals in the non-optimized formula to their representatives that appear in the optimized version. Therefore, the problem of minimizing the resolution tree is to select such paths that use the fewest number of implications in total. The above lemma tells us that paths going to different representatives are disjunct. Therefore, if we can solve the problem for one BIS, then we can solve it for the whole set of literals. The following problem restates the task in graph-theoretical terms for one BIS.

Problem 1. *Let $G \equiv \langle V, E \rangle$ be a directed graph, let $r \in V$ and $S \subseteq V$. Find a set $E' \subseteq E$ such that E' contains a path for each $n \in S$ to r.*

This problem is the directed *Steiner Tree Problem* [38, Definition 1] and is known to be NP-hard [38]. Therefore, we use a simple heuristic based on shortest paths.

Definition 11 (Shortest Path Heuristic). *Given an instance of Problem 1, the* Shortest Paths Heuristic *chooses E' as the union of some shortest paths from the literals in S to their representatives. This is in mathematical notation the following.*

$$E' \stackrel{\text{def}}{=} \{P \mid P \text{ is some shortest path from } l \text{ to } r, \text{ where } l \in S \text{ and } l \mapsto r\} \quad (5.1)$$

To implement the Shortest Paths Heuristic, we just need to refine Swap so it identifies a shortest path from the given literal to its representative and it does

so that the same path is always used for the same literal (see line 2 in SWAP).

> ***Remark 5.22.*** *Since the representatives are chosen at the beginning of the configuration process, the shortest paths for all the literals can be computed once at the beginning and stored.*

The Shortest Paths Heuristic motivates the heuristic we use to choose the representatives. Since the size of the union of the shortest paths is bounded by the sum of the sizes of the individual paths (Equation 5.1), we pick the representative so that the expected distance to other nodes in the BIS is minimal. (A *distance* between two nodes in a graph is the length of a shortest path.)

For a literal l in the BIS S, let d_l denote the sum of distances from all $k \in S$ to l. Note that since any literal is reachable from all the literals in the same BIS, the average path length is $\frac{d_l}{|S|}$.

Recall that each BIS S has a dual BIS S' with reversed edges and negated literals. One of the decisions made earlier about representatives was that l represents S iff $\bar{l}$ represent S'. Therefore, to minimize the average distance to the representative, we look for a literal whose complement performs well too.

Definition 12. *The* Shortest Average Path Heuristics *(*SAP Heuristics*) is that for each BIS S pick as the representative a literal $l \in S$ with the minimal $d_l + d_{\bar{l}}$.*

> ***Remark 5.23.*** *In graph theory, a node minimizing the sum of distances to all other nodes is called* a median of the graph *[150].*

Observation 11. *If the BIS S contains i implications, finding the numbers d_l can be done in $O(|S|\,(|S| + i))$ time and $O(|S|)$ space by repeated breadth first traversals. As a side-effect, this produces shortest path trees that are used later by the optimizer to quickly find the implications that need to be added to the explanation.*

- *Choosing the best representative in general is difficult but it is reasonable to pick such representatives so the average distance to it from a random node in the BIS is short.*

5.3 Summary

The BIS-optimization is based on the fact that the literals in the same strongly connected component of an implication graph are equivalent. This chapter shows how to integrate the BIS-optimization in a configurator that is constructed as described in Chapter 4. The basic idea is to collapse literals in the same strongly connected component of the implication graph, which is constructed upon initialization, and then query the solver for only one literal out of the component; this literal is called the *representative*. The second half of this

chapter investigates how to reconstruct resolution proofs that where obtained from a solver operating on the optimized formula. We show that the choice of representatives may influence the size of the reconstructed resolution tree. Finally, we propose a heuristics for choosing a representative that is aimed at reducing the size of the reconstructed trees.

In summary, the contributions of this chapter are showing how to use the BIS-optimization in interactive configuration and showing how to reconstruct proofs for a non-optimized formula from the pertaining optimized formula. It should be noted that while the BIS-optimization is well known in the SAT community, the proof reconstruction appears to be neglected by the existing research (see Section 10.2).

Chapter 6

Completing a Configuration Process

The text in the previous chapter is concerned the user-support *during* the configuration process. This chapter focuses on its completion. The motivational question is: "Is it necessary for the user to give a value to each and every variable to complete the configuration?"

The chapter starts by a general discussion on configuration completion and considers several different categories of automated support for completion. A concept of *the shopping principle* is introduced (Section 6.1).

The bulk of the chapter is formed by Section 6.2 which studies the implementation of the shopping principle in the context of propositional configuration. It also shows links to concepts known from Artificial Intelligence (Section 6.2.4).

Section 6.4.2 generalizes the shopping principle in the context of general constraints; this will however require a notion of preference. It is shown that the shopping principle is indeed a special case of this general approach. Finally, Section 6.5 summarizes the chapter.

6.1 Completion Scenarios and the Shopping Principle

Recall that a configuration process is complete if all features are bound (Definition 3 and Observation 1). Let us investigate some scenarios of how this can be achieved. We assume that the configuration is carried out with the aid of a backtrack-free configurator (see Chapter 3) and therefore, any bound features are automatically computed after each user decision.

Consider that the user is configuring the feature tree in Figure 6.1 using a

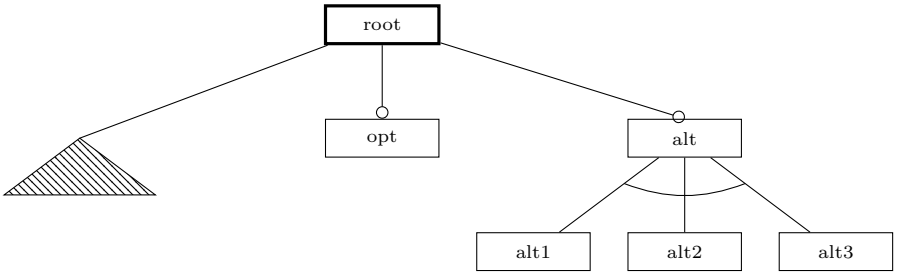


Figure 6.1: Example feature model

backtrack-free configurator. The hatched area represents the features that are already bound, and thus, the only features that are not bound are those that explicitly appear in the figure. Let us assume that the configurator enables the user to indicate that the features in the hatched area are actually those that are interesting for the user and the rest should go away. Upon this indication, the configurator automatically sets all the unbound features to False (eliminates them). Such behavior of the tool would be in accord with the way that people shop at a fruit stall: customers tell a stall-keeper the items they want. They do not specify the items they do not want nor do they need to go over every item in specify for it whether they want it or not.

At a fruit stall this is always possible because there are no dependencies between the items. As we know, in feature models there are dependencies and this mechanism cannot be always applied. Consider, for example, that the user selects the feature `alt` and *then* indicates that he or she is done. The feature `opt` can still be eliminated but one of the `alt1`, `alt2`, or `alt3`, *must* be selected because `alt` requires so. Therefore, if some automated method is to assign values to the unbound features, it must *choose* between these three features. It might not be desirable for the tool to make such choices as it means *making a choice for the user*—it would be overly smart.

> ***Remark 6.24.*** *The issues discussed in the two paragraphs above were already noted by Batory in [15, Section 4.2]. Therefore, this chapter is actually a response to the problem hinted therein.*

The informal discussion above enables us to classify the ways how the user can complete the process.

M (*Manual completion*). The user continues makes decisions up to the point when all considered variables are bound, i.e., each variable has been assigned a value by the user or by a decision inferred by the configurator. The disadvantage of this approach is that the user needs to fill in every single detail and that is cumbersome especially if there are some parts of the problem that

are not of a high relevance to the user. The only assistance the tool provides is the mechanism that infers new decisions or disables some decisions. We will not discuss this case further.

A (*Full blind automation*). The users make decisions up to a point where they believe they made the decisions important for them and invoke a function that automatically computes *some* appropriate values for all the variables that have not been bound yet. The disadvantage of this function is that it takes all the control from the user, in other words, it is making choices for them.

$\mathbf{A}^+$ (*Smart automation*). As in **A**, the users make decisions up to a point where they believe they decided what they needed and invoke a function that is supposed to bind the rest. The function in this scenario, bounds only those variables for which it would not mean making a choice for the user. If there are some variables that cannot be bound like that, the tool highlights them and it is the user who makes that choice.

In some sense, the scenario $\mathbf{A}^+$ is a more careful version of scenario **A**. In both scenarios, the tool helps the user to complete the configuration process. However, scenario **A** binds **all** variables, whereas $\mathbf{A}^+$ binds based on the rule: *the variables that are not bound are eliminated unless that would mean making a choice for the user*. In the remainder of this text it is referred to this rule as the *shopping principle*.

Hence, if **A** is applied to Figure 6.1 after `alt` has been selected, one of the `alt1`, `alt2`, and `alt3` is selected. In contrast to that, if shopping principle is applied in the same situation, the feature `opt` is eliminated, whereas the features `alt1`, `alt2`, and `alt3` are left for the user to choose from.

> ***Remark 6.25.*** *Scenario **A** can be investigated more finely as the binding function can be directed by some cost functions such as minimization of the number of selected features.*

In summary, there are two types of functions that the user may invoke at any step of the process in order to signal that the process should be completed: (**A**) a function that binds all the remaining variables; ($\mathbf{A}^+$) a function that binds only variables for which this would not mean making a choice for the user; we call this function a *shopping principle function*.

- *The shopping principle function eliminates unbound variables but tries not to be overly smart when doing so.*

6.2 Completing a Propositional Configuration Process

Following the motivation above, this section focuses on the completion of a configuration process in terms of propositional logic. The case **A** is straightforward, binding all the remaining unbound variables means finding a solution to the state formula ϕ_i in step i. This is a classical satisfiability problem, which can be solved by a call to a SAT solver or by a traversal of a BDD corresponding to ϕ_i.

The scenario $\mathbf{A}^+$, however, is more intriguing. The shopping principle tells us that what has not been bound should be eliminated, which means setting to FALSE, in terms of propositional logic. However, it is not always possible to set all unbound variables to FALSE. For instance, in $u \vee v$ we cannot set both u and v to FALSE—in this case the user must choose which one should be TRUE. If we consider the formula $x \Rightarrow (y \vee z)$, however, all the variables can be set to FALSE at once and no further input from the user is necessary.

Hence, translating the shopping principle into the language of propositional logic, the objective is to maximize the set of variables set to FALSE without making any choices for the user, i.e., variables that can be *eliminated safely*. Upon a request, the configurator will set the safely-eliminable variables to FALSE and highlight the rest as they need attention from the user. Still, however, we have not established what it formally means to "make a choice for the user"; this is done in the following section.

6.2.1 Eliminating Safely

We begin by a couple of auxiliary definitions that establish what it means for a variable or set of variables to be eliminable. The reference formula, denoted as ϕ, in the following definitions is assumed to be the state formula of the state when the shopping principle function is invoked. We write $\mathcal{V}$ to do note the set of considered variables.

Definition 13 (Eliminable Variable). *For a formula ϕ and a variable $v \in \mathcal{V}$ we write $\mathcal{E}(\phi, v)$ and say that v is* eliminable *in the formula ϕ iff there is a model of ϕ in which the variable v has the value* FALSE.

In plain English, a variable is eliminable if it can be set to FALSE. The following definition generalizes this concept for a set of variables.

Definition 14 (Eliminable Set). *For a formula ϕ and a set of variables $X \subseteq \mathcal{V}$ we write $\mathcal{E}(\phi, X)$ and say that X is* eliminable *in ϕ, iff there is a model of ϕ where* all *the variables in X have the value* FALSE.

The following observations show a connection between satisfiability and eliminable variables.

Observation 12. *A variable v is eliminable in ϕ iff* $\textsc{Sat}(\phi \wedge \neg v)$.

Proof. Any model of ϕ assigning False to v is also a model of $\phi \wedge \neg v$. Hence, if v is eliminable in ϕ, then $\phi \wedge \neg v$ is satisfiable. Conversely, any model M of $\phi \wedge \neg v$ is also a model of ϕ and v has the value False in M. Hence, if $\phi \wedge \neg v$ has at least one model (it is satisfiable), then ϕ has a model with v having the value False. □

Observation 13. *A set X is eliminable in ϕ iff $SAT\left(\phi \wedge \bigwedge_{v \in X} \neg v\right)$.*

Proof. Analogous to the proof for previous observation. □

Example 29. *Let $\phi \stackrel{\text{def}}{=} (u \vee v) \wedge (x \Rightarrow y)$, then the set $\{x, y\}$ is eliminable in ϕ, while the sets $\{u, v\}$ and $\{u, v, x\}$ are not.*

In the example above we see that $\{u, v\}$ is not eliminable but also that $\{u, v, x\}$ is not eliminable. Clearly, if a set is *not* eliminable, it cannot be made eliminable by adding more variables; this property is summarized by the following observation.

Observation 14. *If a set X is* not *eliminable in ϕ then any superset of X is* not *eliminable either.*

The following definition utilizes the concept of eliminable sets to identify variables between which the user must choose.

Definition 15 (choice set). *The set of variables $X \subseteq V$ is a* choice set *in ϕ iff the two following conditions hold.*

1. *X is not eliminable in ϕ, i.e., $\neg\mathcal{E}(\phi, X)$.*
2. *All of the proper subsets of X are eliminable, i.e., $(\forall Y \subsetneq X)(\mathcal{E}(\phi, Y))$.*

Note that the condition *1.* in the above definition ensures that at least one variable in a choice set must be True while the condition *2.* ensures that the set is as "tight" as possible—removing any variable yields a set where all variables can be set to False at once.

> ***Remark 6.26.*** *The requirement for tightness of a choice set comes from Observation 14: if it was only required that it is not possible to eliminate the set X, any of its supersets would have the same property and thus taking on unrelated variables.*

- *Variables between which the user must choose form minimal sets of variables where at least one variable must be* True.

The concept of choice enables us to define those variables that can be eliminated without making a choice for the user; these variables are called dispensable.

Definition 16 (Dispensable variables). *A variable $v \in \mathcal{V}$ is* dispensable *in a formula ϕ iff it does not belong to any choice set in ϕ. We write Υ_ϕ to denote the set of all dispensable variables in ϕ.*

Example 30. *Let $\phi \stackrel{\text{def}}{=} (u \vee v) \wedge (x \Rightarrow y)$. All the variables u, v, x, and y are eliminable. The set $\{u, v\}$ is a choice set because it cannot be eliminated while each of its subsets is eliminable.*

The only sets that are not *eliminable and contain x or y must also contain u and v, i.e., they are $\{u, v, x\}$, $\{u, v, y\}$, and $\{u, v, x, y\}$. However, none of these sets is a choice set because it does not fulfill the tightness requirement as $\{u, v\}$ is not eliminable.*

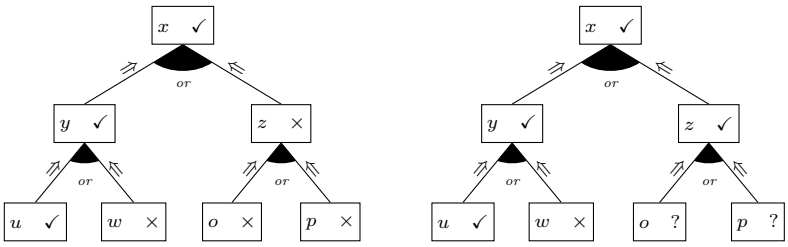
Hence $\Upsilon_\phi = \{x, y\}$, i.e., the variables x and y are dispensable while the variables u and v are not (though they are eliminable).

> ***Remark 6.27.*** *Dispensability treats* True *and* False *asymmetrically. In the formula $x \Rightarrow y$, it is not possible to eliminate y without eliminating x, i.e., eliminating y forces x to* False*. This, however, does not collide with dispensability because both x and y can be eliminated at the same time.*
> *In contrast, eliminating u in $u \vee v$ forces v to* True *and therefore the set $\{u, v\}$ is not eliminable and the variables u and v are not dispensable.*

> ***Remark 6.28.*** *In any formula of the form $x_1 \vee \cdots \vee x_n$ the set $\{x_1, \ldots, x_n\}$ is a choice set because it is not eliminable but any of its proper subsets is because it is sufficient for only one of the $x_1, \ldots, x_n$ to be* True*. Therefore, none of the variables $x_1, \ldots, x_n$ are dispensable.*

The following example illustrates how dispensable variables are applied in the context of a configurator process.

Example 31. *Let ϕ_0 be defined as in the previous example: $\phi_0 \stackrel{\text{def}}{=} (u \vee v) \wedge (x \Rightarrow y)$. The user invokes the shopping principle function. Since x and y are dispensable, both are eliminated (set to* False*). The variables u and v are highlighted as they need user's attention. The user selects u, which results in the formula $\phi_1 \stackrel{\text{def}}{=} \phi \wedge u = u \wedge (x \Rightarrow y)$. The variable v becomes dispensable and can be eliminated automatically. The configuration process is complete as all variables are bound: x, y, and v are bound to* False*, u to* True*.*



(a) x, y, and u selected; the rest are dispensable

(b) x, y, u, and z selected; only w dispensable

Figure 6.2: Examples of situations with selected and dispensable variables. Selected variables are marked with ✓; dispensable variables are marked with ×; other variables are marked with ? .

Example 32. *Figure 6.2 shows two incomplete configurations of a feature diagram. The examples are to show that the definition of dispensable variables is in accord with the intuition behind feature diagrams. In Figure 6.2a, w is dispensable because u is selected. Similarly, z is dispensable because y is selected. More generally, if one variable from an or-group is selected, then all the other variables in that group are dispensable.*

Since z is dispensable, the variables o and p are dispensable. More generally, if a variable is dispensable, then the subtree rooted in that variable becomes dispensable, provided there are no constraints cross-cutting the tree hierarchy.

The situation is slightly different in Figure 6.2b where z is selected. Now the semantics tells us that $z \wedge (z \Rightarrow (o \vee p))$ and therefore o and p cannot be eliminated, while each one separately is eliminable. Hence, $\{o, p\}$ forms a choice set, and, o and p are not dispensable. More generally, whenever a variable is selected, the or-groups belonging to it become choice sets.

6.2.2 Eliminating Dispensable Variables

The shopping principle function is supposed to eliminate all dispensable variables. The definition of dispensability, however, does not directly say that this is actually possible. Here we show that the set of dispensable variables is indeed eliminable. The proof of such will require the following lemma.

Lemma 1. *Let ϕ be a formula and X be a finite set of variables not eliminable in ϕ, then there is a subset of X that is a choice set.*

Proof. If all proper subsets of X are eliminable, then we are done because X is a choice set as it is not eliminable while each of its subset is eliminable. If X contains a proper subset $X' \subsetneq X$ that is *not* eliminable then we repeat with X'. This process must eventually terminate because X is finite. □

Now it is straightforward to show that dispensable variables can be eliminated all at once.

Proposition 6. *For a satisfiable ϕ, the set of dispensable variables is eliminable in ϕ, i.e., $\mathcal{E}(\phi, \Upsilon_\phi)$.*

Proof. (by contradiction) Let us assume that Υ_ϕ is *not* eliminable then it must contain a set X that must be decided due to Lemma 1. Since Υ_ϕ consists only of dispensable variables, which cannot be members of any choice set, the set X must be empty. But if the empty set is a choice set, then it must not be eliminable. This is a contradiction to the fact that ϕ is satisfiable because the empty set is eliminable iff ϕ is satisfiable due to Observation 13. □

- *All the dispensable variables can be eliminated all at once.*

6.2.3 Dispensable Variables and Minimal Models

> ***Remark 6.29.*** *This part of the dissertation relies mostly on the concepts and notation introduced in Section 2.1.2. In particular that $\mathcal{M}(\phi)$ denotes the set of minimal models of ϕ and models of formulas are treated as sets of variables with the value* True.

This section shows a relation of dispensable variables to minimal models (Section 2.1.2). This relation is used later on to compute dispensable variables. We start with a lemma relating minimal models and eliminable sets.

Lemma 2. *If a set of variables is eliminable, then there is a minimal model with all these variables assigned to* False. *Equivalently, using the notation introduced so far: if $\mathcal{E}(\phi, X)$ then there is a minimal model M of ϕ such that $M \cap X = \emptyset$.*

Proof. Let M be a model of ϕ such that it assigns False to all $x \in X$, i.e., $M \cap X = \emptyset$; such model exists because $\mathcal{E}(\phi, X)$ holds. If M is minimal, then we are done. If M is not minimal, then there is a model M_1 of ϕ smaller than M, i.e., $M_1 \subsetneq M$. Because M_1 was obtained from M by removing some variables, it still holds that $M_1 \cap X = \emptyset$. Therefore, the process can be repeated with M_1. Since the initial model M is finite then we must eventually find a minimal model by repeating the process. □

Claim 6. *A variable v is dispensable iff it is* False *in all minimal models. In notation: $\phi \models_{\min} \neg v$.*

Proof. To show that if v is dispensable then it is False in all minimal models we prove the contrapositive: if there is a minimal model M assigning True to v, then v is not dispensable. Let Y be the set of variables that are False in M

and let $X \stackrel{\text{def}}{=} Y \cup \{v\}$. The set X is not eliminable because any model assigning FALSE to all variables in X would be smaller than M but M is minimal. Since X is not eliminable, from Lemma 1, there must be a set $X' \subseteq X$ that is a choice set. Since the set $X' \subseteq Y \cup \{v\}$ and Y is eliminable and X' is not, the set X' must contain the variable v. Hence, v is not dispensable as it belongs to a choice set.

To prove the inverse implication, we prove by contradiction that if v is FALSE in all minimal models, then it must be dispensable. Let v be *not* dispensable in ϕ and $\phi \models_{\min} \neg v$. Because v is not dispensable there must be a set X such that $v \in X$, $\neg\mathcal{E}(\phi, X)$, and $\mathcal{E}(\phi, Z)$ for any $Z \subsetneq X$ (see definitions 15, 16). Let $Y \stackrel{\text{def}}{=} X \setminus \{v\}$, since Y is a proper subset of X, then $\mathcal{E}(\phi, Y)$. From Lemma 2, there is a minimal model M that assigns FALSE to all variables in Y. Because v is FALSE in *all* minimal models of ϕ, it must be FALSE in M as well. This contradicts the assumption that X is not eliminable because the model M shows that the set $X = (X \setminus \{v\}) \cup \{v\}$ is eliminable. □

Example 33. *Let $\phi_0 \stackrel{\text{def}}{=} (u \vee v) \wedge (x \Rightarrow y)$. The minimal models of the formula ϕ_0 are $\{u\}$, $\{v\}$, hence $\phi_0 \models_{\min} \neg x$ and $\phi_0 \models_{\min} \neg y$. Then, if the user invokes the shopping principle function, x and y are eliminated, i.e., $\phi_1 \stackrel{\text{def}}{=} \phi_0 \wedge \neg x \wedge \neg y$. And, the user is asked to resolve the competition between $u \vee v$, selects u, resulting in the formula $\phi_2 \stackrel{\text{def}}{=} \phi_1 \wedge u$ with the models $\{u\}$ and $\{u, v\}$ where only the model $\{u\}$ is minimal hence v is set to* FALSE *as dispensable. The configuration process ends because u has the value* TRUE *and the rest are dispensable.*

6.2.4 Dispensable Variables and Non-monotonic Reasoning

This section investigates the relation between dispensable variables and a subdomain of Artificial Intelligence known as non-monotonic reasoning. This relationship is not only of a theoretical significance but also of a practical one as it opens other ways of computing dispensable variables (other than those that are shown in this dissertation).

The first thing to be clarified is the term *non-monotonic* reasoning appearing in the title of this section. Unlike in classic logic inference, in *non-monotonic* reasoning an increase in knowledge does not necessarily mean an increase in inferred facts. Non-monotonicity is observed on the set of dispensable variables as well: adding more constraints neither implies that the set of dispensable variables expands nor does it imply that it shrinks. The following example illustrates this property.

Example 34. *Consider $\phi_0 \stackrel{\text{def}}{=} x \Rightarrow (y \vee z)$. The only minimal model of ϕ is $\emptyset$,*

which assigns FALSE *to all the three variables and therefore, all of them are dispensable due to Claim 6. Using the* $\models_{\min}$ *notation, we write* $\phi_0 \models_{\min} \neg x$, $\phi_0 \models_{\min} \neg y$, *and* $\phi_0 \models_{\min} \neg z$.

Conjoining ϕ_0 *with* x *yields* $\phi_1 \stackrel{\text{def}}{=} (x \Rightarrow (y \vee z)) \wedge x$ *with the minimal models* $\{x, y\}$ *and* $\{x, z\}$. *Neither of the variables is dispensable now, which means that the set of dispensable variables has shrunk when going from* ϕ *to* ϕ_1. *In terms of the relation* $\models_{\min}$, *it is no longer possible to infer what was possible before, i.e.,* $\phi_1 \not\models_{\min} \neg y$, $\phi_1 \not\models_{\min} \neg z$, *and* $\phi_1 \not\models_{\min} \neg x$. *Moreover, in the case of variable* x *we obtain a fact contradictory to the previous one, i.e.,* $\phi_1 \models_{\min} x$.

Conjoining ϕ_1 *with the formula* y *yields* $\phi_2 \stackrel{\text{def}}{=} (x \Rightarrow (y \vee z)) \wedge x \wedge y$, *which has the only minimal model* $\{x, y\}$ *and the variable* z *becomes dispensable. Hence, the set of dispensable variables has expanded when going from* ϕ_1 *to* ϕ_2.

The relation $\cdot \models_{\min} \cdot$ is closely tied to the concept of *circumscription* introduced by McCarthy in the 60's as a form of non-monotonic reasoning [143]. While McCarthy originally operated on first-order logic, there is also the concept of *propositional circumscription*, which is closer to the topic of this dissertation (see for example [36]).

A circumscription of a propositional formula ϕ is the set of all minimal models of ϕ and reasoning with circumscription means determining which statements hold in all minimal models of a formula, i.e., determining the validity of $\phi \models_{\min} \psi$. Hence, to determine whether a variable is dispensable is in fact a special case of a circumscription reasoning due to Claim 6.

> ***Remark 6.30.*** *Observe that the set of* all *models does shrink when the formula is constrained while the set of* minimal *models may shrink or expand. Since the classic relation* $\phi \models \psi$ *means that* ψ *is* TRUE *on all models while* $\phi \models_{\min} \psi$ *means that* ψ *is* TRUE *in all* minimal *models of* ϕ, *the relation* $\cdot \models_{\min} \psi$ *behaves non-monotonically and* $\cdot \models \psi$ *monotonically.*

Another term related to circumscription is the *Closed World Assumption (CWA)* coming from the field of logic programming and knowledge representation. Roughly speaking, CWA is based on the idea that a fact should not be inferred unless there is a justification for it.

The literature offers several definitions of CWA [36, 63]. The definition most relevant to this dissertation is the one of the *Generalized Closed World Assumption (GCWA)* introduced by Minker [152] (also see [36, Definition 1]). The following two definitions introduce GCWA using the notation of this dissertation.

Definition 17 (free of negation). *A variable* v *is* free of negation *in the*

formula ϕ iff for any positive clause B for which $\phi \not\models B$, it holds that $\phi \not\models v \vee B$. A clause is positive if it contains only positive literals, i.e., it is in the form $x_1 \vee \cdots \vee x_n$ where $x_1, \ldots, x_n$ are variables.

Definition 18 (GCWA). *The* closure *$C(\phi)$ of ϕ w.r.t. GCWA is defined as*

$$C(\phi) \stackrel{\text{def}}{=} \phi \cup \{\neg v \mid v \text{ is free for negation in } \phi\}$$

The formula ψ holds in the formula ϕ under GCWA iff $C(\phi) \models \psi$.

The following lemma rephrases the condition for free of negation in terms of eliminable sets.

Lemma 3. *A variable v is free of negation in ϕ iff it satisfies the condition: if $\mathcal{E}(\phi, X_B)$ then $\mathcal{E}(\phi, X_B \cup \{x\})$ for any set of variables X_B.*

Proof. First observe that $\phi \not\models \psi$ iff $\phi \wedge \neg\psi$ is satisfiable. This lets us rewrite the definition "v is free of negation" as follows. If $\phi \wedge \bigwedge_{x \in V_B} \neg x$ is satisfiable, then $\phi \wedge \neg v \wedge \bigwedge_{x \in V_B} \neg x$ is satisfiable. This can be further rewritten using the concept of eliminable sets as: if $\mathcal{E}(\phi, X_B)$ then $\mathcal{E}(\phi, X_B \cup \{x\})$ (see Observation 13). □

The following claim relates dispensable variables to those that are free of negation.

Claim 7. *A variable v is dispensable in ϕ iff it is free of negation in ϕ.*

Proof. Using Lemma 3 we show that v is dispensable iff it holds that if $\mathcal{E}(\phi, X)$ then $\mathcal{E}(\phi, X \cup \{x\})$.

To show that if v is free of negation then it must be dispensable, we prove the contrapositive, i.e., if v is not dispensable then it is not free of negation. If the variable v is not dispensable, then it belongs to some choice set Y, in particular it holds $\neg\mathcal{E}(\phi, Y)$ and $\mathcal{E}(\phi, Y \setminus \{v\})$ and therefore v is not free of negation because the condition is violated for the set $X \stackrel{\text{def}}{=} Y \setminus \{v\}$.

Conversely, if v is not free of negation, then there is a set X for which $\mathcal{E}(\phi, X)$ but not $\mathcal{E}(\phi, X \cup \{v\})$. Since $X \cup \{v\}$ is not eliminable, then there is a set X' that is a subset of $X \cup \{v\}$ and it is a choice set (Lemma 1). Because X itself is eliminable in ϕ, the set X' must contain the variable v and therefore v is not dispensable. □

The following claim relies on research on circumscription done by Eiter and Gottlob in order to show the complexity classification of computation of dispensable variables.

Claim 8. *To determine whether a variable is dispensable in the formula ϕ is Π_2^P. This holds even if ϕ in CNF.*

Proof. From Claim 6, v is dispensable iff $\phi \models_{\min} \neg v$. See [63, Lemma 3.1] for Π_2^P completeness of $\phi \models_{\min} \neg v$. □

6.3 Computing Dispensable Variables

Claim 6 gives us a way to compute dispensable variables from minimal models. The claim tells us that a dispensable variable must be False in all minimal models. If we have the set of all minimal models at our disposal, we can go through all of them and mark all the variables having the value True in any of them as non-dispensable. The function Dispensable in Figure 6.3 realizes this idea.

```
Dispensable(φ : Formula, 𝒱: Set of Variable) : Set of Variable
1  ℳ ← MinModels(φ)
2  R ← 𝒱
3  foreach M ∈ ℳ
4        do R ← R ∖ M
5  return R
```

Figure 6.3: Computing dispensable variables from minimal models

The procedure Dispensable initially assumes that all variables are dispensable (line 2) and then it iterates over all minimal models. If a minimal model contains a variable x, i.e., it assigns True to x, then x is not dispensable (line 4). Once the loop terminates, the set R contains only the dispensable variables.

What is missing at this point is an implementation of the function MinModels that computes the set of all minimal models.

The algorithm used in this section to collect all minimal models is based on the following loop: (1) Find some minimal model of the formula. (2) Disable from the formula all models greater or equal to that minimal model. (3) If the formula is still satisfiable then goto (1).

The step (2) shrinks the set of models of the formula in such way that the model found in step (1) is the only minimal model taken out. Since every formula has a finite number of minimal models eventually all of them are taken out, the formula becomes unsatisfiable and the loop terminates. Figure 6.4 presents pseudo-code for an implementation of the function MinModels realizing this idea.

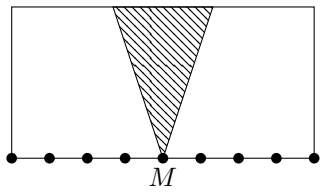
The heart of MinModels is the loop on lines 7–9. At the beginning of each iteration it first records the minimal model found last (line 7), then it finds a minimal model M (line 8) and strengthens the formula such that all models

MinModels(ϕ_0 : *Formula*) : Set of *Model*

```
1  R ← ∅
2  φ ← φ0
3  M ← MinModel(φ)
4  ▷ invariant: R ∪ M(φ) = M(φ0)
5  ▷ invariant: R ∩ M(φ) = ∅
6  while M ≠ null
7        do R ← R ∪ {M}
8           M ← MinModel(φ)
9           φ ← φ ∧ ⋁_{v∈M} ¬v
10 return R
```

Figure 6.4: Computing the set of minimal models

greater or equal to M are no longer models of ϕ (line 9).



The little circles at the bottom represent minimal models. The hatched area represents the models removed from the formula along with the minimal model M.

Figure 6.5: Removal of models

The effect of line 9 is illustrated by Figure 6.5. The hatched area represents the models removed by the strengthening. The models in this area must assign True to all the variables that have the value True in M. Therefore the hatched area corresponds to the formula $\bigwedge_{v\in M} v$. Since this area is to be *excluded*, ϕ is conjoined with the *negation* of this formula which is $\bigvee_{v\in M} \neg v$.

Example 35. *Let $\phi_0 \stackrel{\text{def}}{=} x \vee (y \wedge z)$ (the subscript in ϕ_i represents the different iterations of the loop in* MinModels*). Let $\{x\}$ be the first minimal model found, then $\phi_1 \stackrel{\text{def}}{=} \phi_0 \wedge \neg x$ according to line 9. The formula ϕ_1 can be rewritten equivalently as $y \wedge z \wedge \neg x$, which has a single minimal model $\{y, z\}$. Finally, executing line 9 yields $(y \wedge z \wedge \neg x) \wedge (\neg y \wedge \neg z)$, which has no (minimal) models and the algorithm terminates.*

Line line 9 of MinModels requires a function MinModel that computes some minimal model of ϕ. To compute a minimal model we utilize a SAT solver, which is described by Section 6.3.1. First, however, let us look more closely at the properties of the procedure MinModels driving the process. The first observation is concerned with the form of the formula guaranteeing that a SAT solver can always be invoked.

Observation 15. *If the input formula ϕ_0 is in CNF, then ϕ is in CNF at all times.*

Proof. The initial formula ϕ_0 is in CNF, and ϕ is conjoined with $\bigvee_{v \in M} \neg v$ which is a clause. □

Observation 16. *Line 9 removes from ϕ only models greater or equal to M. Formally, let ϕ and ϕ' denote the formulas before and after performing line 9, respectively. Let M be the model found in line 8 and M' be some model of ϕ. Then M' is not a model of ϕ' iff $M \subseteq M'$.*

Proof. If M' is a model of ϕ but is not a model of ϕ', it must violate the newly-added clause $\bigvee_{v \in M} \neg v$. That means that all the $v \in M$ are TRUE in M', which is equivalent to $M \subseteq M'$. Conversely, if $M \subseteq M'$ then M' is violating the newly-added clause $\bigvee_{v \in M} \neg v$ and therefore it is not a model of ϕ'. □

Lemma 4. *Line 9 removes one minimal model, which is M. Formally, let ϕ and ϕ' denote the formulas before and after performing line 9, respectively. Then, $\mathcal{M}(\phi') = \mathcal{M}(\phi) \setminus \{M\}$.*

Proof. We start by proving $(\mathcal{M}(\phi) \setminus \{M\}) \subseteq \mathcal{M}(\phi')$ by showing that any minimal model N of ϕ that is not equal to M, is a minimal model of ϕ'. Firstly, N must not be a superset of M because it is a minimal model in ϕ. From which it follows that N is a model of ϕ' due to Observation 16. To show that N is minimal model of ϕ' by contradiction we assume there is a model N' of ϕ' such that $N' \subseteq N$. Since N is minimal in ϕ, the set N' must not be a model of ϕ but that contradicts Observation 16 as models of ϕ' are a subset of the models of ϕ according to that observation.

Conversely we prove $\mathcal{M}(\phi') \subseteq \mathcal{M}(\phi) \setminus \{M\}$ by showing that any minimal model of ϕ' is a minimal model of ϕ. Let $N \in \mathcal{M}(\phi')$, then N is a model of ϕ as ϕ' was obtained by strengthening ϕ. For contradiction, let us assume that N is *not* a minimal model of ϕ. Then there is model N' of ϕ such that $N' \subsetneq N$ and N' is not a model of ϕ'. Since N' is not a model of ϕ' but is a model of ϕ, it must be $M \subseteq N'$ due to Observation 16. Hence, we have $M \subseteq N'$ and $N' \subseteq N$. From transitivity of $\subseteq$ we obtain $M \subseteq N$ but that is a contradiction because N is a model of ϕ', which does not have models that are supersets of M due to Observation 16. □

Lemma 5. *The loop invariants in* MINMODELS *on lines 4 and 5 hold.*

Proof. The invariants $R \cup \mathcal{M}(\phi) = \mathcal{M}(\phi_0)$ and $R \cap \mathcal{M}(\phi) = \emptyset$ are trivially established upon the entry of the loop as R is empty and ϕ is equal to ϕ_0.

Let ϕ and R denote the values at the beginning of an arbitrary iteration of the loop and ϕ' and R' at the end of the iteration. Due to Lemma 4, $\mathcal{M}(\phi') = \mathcal{M}(\phi) \setminus \{M\}$, which gives us the following equalities that prove the first invariant ($\mathcal{M}(\phi_0) = R' \cup \mathcal{M}(\phi')$).

$$\begin{aligned}
&\mathcal{M}(\phi_0) && =^{\text{induction hypothesis}} \\
&R \cup \mathcal{M}(\phi) && =^{\text{equivalence}} \\
&(R \cup \{M\}) \cup (\mathcal{M}(\phi) \setminus \{M\}) && =^{\text{Lemma 4 and equivalence}} \\
&R' \cup \mathcal{M}(\phi')
\end{aligned}$$

The second invariant is clearly preserved by the loop since $\mathcal{M}(\phi)$ and R are disjoint and therefore $\mathcal{M}(\phi')$ and R' are disjoint as well. □

Claim 9. *When the algorithm* MinModels *terminates, it returns all minimal models of the input formula ϕ_0.*

Proof. Since the invariant $\mathcal{M}(\phi) \cup R = \mathcal{M}(\phi_0)$ holds and when the loop terminates $\mathcal{M}(\phi)$ is empty, $R = \mathcal{M}(\phi_0)$ □

Claim 10. *The algorithm* MinModels *is terminating. In particular it iterates as many times as many minimal models there are in the input formula ϕ_0.*

Proof. From Lemma 4 we know that the loop decreases the number of minimal models of ϕ by one. As the loops stops when ϕ has no minimal models, the loop iterates as many times as many minimal models ϕ_0 has. □

6.3.1 Computing a Minimal Model with a SAT Solver

As shown in the previous section, we can enumerate the set of all minimal models if we have the means to compute *one* minimal model. This section shows that a SAT solver that is based on the principles outlined in Section 2.3 can be easily modified to find a minimal model. Intuitively, the modification is to enforce the preference of False over True as specified by the following definition.

Note that the term decision in this section refers to the decisions that the solver makes to traverse the search space not the decisions that the user makes to reach the current state formula.

Definition 19. *We say that a SAT solver is* biased to False *iff the solver always makes the decision $\neg v$ before the decision v for each variable v.*

Example 36. *Let $\phi \stackrel{\text{def}}{=} x \vee y \vee z$ be a formula given to a solver biased to* False*. The solver makes the decision $\neg x$ followed by the decision $\neg y$. At this point unit propagation assigns* True *to z, resulting in the satisfying assignment $\neg x, \neg y, z$.*

Note that for any ordering of decisions a solver biased to False *assigns* False *to two of the variables and* True *to the remaining one.*

The example above illustrates that solvers biased to False tend to find solutions with as many False values as possible, roughly speaking. In fact, we will see that they always return minimal models. To show that, we first make an observation that if a biased solver ever assigns True to a variable, it must do so unless some of the previous assignments are changed.

Lemma 6. *Let ϕ be a satisfiable CNF formula to which a solution was found by a SAT solver biased to* False*. Let $l_1, \ldots, l_n$ be literals corresponding to the assignments made by the solver (l_1 is the first decision and l_n is the last), then for any assignment for which $l_i \equiv v_i$ it holds $\phi \models (l_1 \wedge \cdots \wedge l_{i-1}) \Rightarrow l_i$.*

Proof. There are two ways how the solver can assign a value to a variable: either in unit propagation or by a decision.

If the assignment $l_i \equiv v_i$ was made by unit propagation then there exists a clause $k_1 \vee \cdots \vee k_l \vee v_i$ such that the literals $k_1, \ldots, k_l$ evaluate to False under the assignment $l_1, \ldots, l_{i-1}$. Hence, v_i must have the value True.

If the solver made the decision $l_i \equiv v_i$, then the formula must be unsatisfiable under the assignment $l_1, \ldots, l_{i-1}, \neg v_i$ due to Property 1[1] and due to the fact that the solver must have tried the decision $\neg v_i$ before as it is biased to False. □

Proposition 7. *Let ϕ be a satisfiable CNF formula to which a solution was found by a SAT solver biased to* False*. Let M be a satisfying assignment returned by the solver represented by the literals $l_1, \ldots, l_n$ sequenced in the order as the assignments were made by the solver, then M assignment is a minimal model of ϕ.*

Proof (by contradiction). Let M' be an assignment assigning False to some variables to which M assigns the value True thus showing that the model M is not minimal. Let v_k be the variable from these variables assigned the first, i.e., M assigns True to v_k and M' assigns False and there is no such other variable assigned earlier. Since M' agrees with M on the assignments $l_1, \ldots, l_{k-1}$, due to Lemma 6 the assignment M' cannot be a model of ϕ. □

- *Using a SAT solver it is possible to enumerate all minimal models and thus identify the dispensable variables.*

[1] The property tells us that once the solver assigned values to some variables, and, this assignment is possible to complete into a satisfying assignment, such assignment will be returned by the solver (see Section 2.3).

6.4 Completing Configuration Processes for General Constraints

The previous section investigates how to help a user with the completion of a configuration of *propositional* constraints, motivated by the shopping principle. This section investigates how to generalize the principle for non-propositional constraints.

The shopping principle function attempts to eliminate as many variables as possible. This can be alternatively seen as that the user *prefers* the unbound variables to be False (see also Remark 6.27). This perspective helps us to generalize the approach to the case of non-propositional constraints under the assumption that there is some notion of preference between the solutions.

First, let us establish the principles for preference that are assumed for this section. (1) It is a partial order on the set in question. (2) It is static in the sense that all users of the system agree on it, e.g., it is better to be healthy and rich than sick and poor. (3) If two elements are incomparable according to the ordering, the automated support shall not decide between them, instead the user shall be prompted to resolve it.

6.4.1 Concepts for Non-propositional Configuration

The following definitions enable us to discuss the general case formally. We start by a general definition of the instance to be configured, i.e., the initial input to the configurator, corresponding to the set of possibilities that the user can potentially reach. As in the propositional case, the user can make decision about variables and the configuration is complete when all variables' values are determined. The difference is, however, that a variable can have a value from a larger domain than just True and False. The following definition realizes this intuition in mathematical terms (this definition is inspired by [101] and by constraint satisfaction problem, see Section 2.1.4).

Definition 20 (Solution Domain). *A* Solution Domain *is a triple* $\langle \mathcal{V}, \mathcal{D}, \phi \rangle$ *where* $\mathcal{V}$ *is a set of variables* $\mathcal{V} = \{v_1, \ldots, v_n\}$, $\mathcal{D}$ *is a set of respective domains* $\mathcal{D} = \{D_1, \ldots, D_n\}$, *and the constraint* $\phi \subseteq D_1 \times \cdots \times D_n$ *is an n-ary relation on the domains.*

A variable assignment *is an n-tuple* $\langle c_1, \ldots, c_n \rangle$ *from the Cartesian product* $D_1 \times \cdots \times D_n$, *where the constant* c_i *determines the value of the variable* v_i *for* $i \in 1 \ldots n$. *For a constraint* ψ, *a variable assignment* α *is a* solution *iff it satisfies the constraint, i.e.,* $\alpha \in \psi$.

Example 37. *Let us have two hardware components with the respective con-*

stant prices p_1, p_2 and a configurable maximum price. This can be modeled as the following solution domain.

Let $\mathcal{V} \stackrel{\text{def}}{=} \{c_1, c_2, p\}$, $\mathcal{D} \stackrel{\text{def}}{=} \{\{0,1\}, \{0,1\}, \mathbb{N}\}$, and the relation

$$\phi \stackrel{\text{def}}{=} \{\langle s_1,\ s_2,\ k\rangle \mid (p_1 \cdot s_1 + p_2 \cdot s_2) \leq k\}$$

The solution domain expresses that the two components can be either included or not and that the price of the included component(s) must be less than the maximum price, which is a natural number.

It is easy to see that a propositional formula is expressible as a solution domain by representing satisfying assignments of the formula by the solutions of the solution domain.

Observation 17. *Let ϕ be a propositional formula on a finite set of variables $\mathcal{V} \equiv \{v_1, \ldots, v_n\}$. Let $D_i \stackrel{\text{def}}{=} \{\textsc{False}, \textsc{True}\}$ for $i \in 1 \ldots n$. Let ψ be relation on $D_1 \times \cdots \times D_n$ defined as follows.*

$$\psi \stackrel{\text{def}}{=} \{\langle b_1, \ldots, b_n\rangle \mid \begin{array}{l} \text{the assignment assigning } b_i \text{ to } v_i \\ \text{for } i \in 1..n \text{ is a satisfying assignment of } \phi\} \end{array}$$

Then $\langle \mathcal{V}, \{D_1 \ldots D_n\}, \psi\rangle$ is a solution domain such that there is a bijection between its solutions and the satisfying assignments of ϕ.

A configuration process of a solution domain $\langle \mathcal{V}, \mathcal{D}, \phi\rangle$ is defined in a fashion analogous to the propositional case. The user actions are making a decision or retracting one. A decision is an arbitrary constraint on the solution domain being configured. In particular, if a decision ξ_j is of the form $v_i = c$ for a variable v_i and a constant $c \in D_i$, we say that the variable v_i has been assigned the value c in step j.

The configuration process is represented as a sequence of sets of decisions such that the first set is empty and subsequent ones are obtained by adding or removing a decision. If Γ_i is the set of decisions in step i, the *state constraint* is the set of solutions determined by the decisions, i.e., $\phi \cap \bigcap_{\xi \in \Gamma_i} \xi$. A variable v is *bound to the constant c* in the state constraint ϕ_i iff all the solutions of ϕ_i assign the value c to v. A configuration process is complete iff all the variables in $\mathcal{V}$ are bound.

Note that the propositional configuration process is a special case of the general one. In particular, selecting a variable means adding the constraint $x = \textsc{True}$ and eliminating a variable means adding $x = \textsc{False}$.

A backtrack-free configurator in this process disallows assigning values that do not appear in any of the solutions of the current state constraint. More specifically, in step l the configurator disallows all values $c \in D_i$ of the variable v_i for

which there is no solution of the state constraint ϕ_l of the form $\langle c_1, \ldots, c, \ldots c_n \rangle$. For convenience, if the variable v_i is bound to some value c (all values but c are disallowed for the domain D_i), then the configurator automatically assigns c to the variable v_i.

6.4.2 Configuration with Preference

Since this section requires the notion of preference, the definition of a solution domain is extended to accommodate for such.

Definition 21 (Ordered Solution Domain). *An* Ordered Solution Domain (OSD) *is a quadruple* $\langle \mathcal{V}, \mathcal{D}, \phi, \prec \rangle$ *where* $\langle \mathcal{V}, \mathcal{D}, \phi \rangle$ *is a solution domain and* $\prec$ *is a partial order on the Cartesian product* $D_1 \times \cdots \times D_n$*. For a constraint* ϕ*, a solution* α *is* optimal *iff there is no solution* α' *of* ϕ *such that* $\alpha' \neq \alpha$ *and* $\alpha' \prec \alpha$*.*

Since an ordered solution domain is an extension of a solution domain, the configuration concepts apply here as well. However, the extra information can be exploited in order to help the user with completing the configuration process. Let us assume that a user is configuring an ordered solution domain and we wish to provide assistance with configuring variables that have lesser importance for the user, similarly as we did with the shopping principle. Just as before, the configuration proceeds as normal up to the point where the user configured all those variables the user wanted to configure and then invokes a function that tries to automatically configure the unbound variables using the given preference.

The assumption we make here is that the variables that were not given a value yet should be configured such that the result is optimal while preserving the constraints given by the user so far. Since the preference relation is a partial order, there may be multiple optimal solutions. As we do not want to make a choice for the users, we let them focus only on optimal solutions.

If non-optimal solutions are ignored, the configurator can identify such values that never appear in any optimal solution. Dually, the configurator identifies values that appear in all optimal solutions, the following definitions establish these concepts.

Definition 22 (Non-Optimal Values). *For a constraint* ϕ *and a variable* v_i*, a value* $c \in D_i$ *is* non-optimal *iff the variable* v_i *has the value* c *only in* non-optimal *solutions of* ϕ*. Equivalently, the variable* v_i *has a value different from* c *in all optimal solutions of* ϕ*.*

Definition 23 (Settled Values and Variables). *For a constraint* ϕ *and a variable* v_i*, a value* $c \in D_i$ *is* settled *iff* v_i *has the value* c *in all optimal solutions*

of ϕ. A variable v_i is settled iff there is some settled value of v_i.

Non-optimal and settled values are related by the following observation.

Observation 18. *For a constraint ϕ and a variable v_i, a value $c \in D_i$ is settled in ϕ iff all values $c' \in D_i$ different from c are non-optimal in ϕ.*

Example 38. *Let $x, y, z \in \{0, 1\}$. Consider a constraint requiring that at least one of x,y,z is set to 1 (is selected). And the preference relation will express that we prefer lighter and cheaper solutions where x, y, and z contribute to the total weight by 1, 2, 3 and to the total price by 10, 5, and 20, respectively. Hence, the solutions satisfy $(x + y + z > 0)$, and $\langle x_1, y_1, z_1\rangle \prec \langle x_2, y_2, z_2\rangle$ iff $(10x_1+5y_1+20z_1 \leq 10x_2+5y_2+20z_2) \wedge (1x_1+2y_1+3z_1 \leq 1x_2+2y_2+3z_2)$. Any solution setting z to 1 is non-optimal as z is more expensive and heavier than both x and y, and hence the configurator sets z to 0 (it is settled). Choosing between x and y, however, needs to be left up to the user because x is lighter than y but more expensive than y.*

As noted in Observation 17, propositional configuration, can be seen as a special case of a configuration of a solution domain with the variable domains $\{\textsc{True}, \textsc{False}\}$. The following observation relates settled and dispensable variables (definition 23 and 16).

Observation 19. *Let ϕ be a propositional formula and $\langle \psi, \mathcal{V}, \mathcal{D}\rangle$ be a corresponding solution domain (see Observation 17), and let $\langle b_1, \ldots, b_n\rangle \prec \langle b'_1, \ldots, b'_n\rangle \stackrel{\text{def}}{=} \bigwedge_{i\in 1..n} b_i \Rightarrow b'_i$. Then, a variable is dispensable in ϕ iff the value* False *is settled in the ordered solution domain $\langle \psi, \mathcal{V}, \mathcal{D}, \prec\rangle$.*

Proof. We note that when $\alpha \prec \beta$ then if α assigns True to any variable, β must do so as well. Hence, the ordering $\cdot \prec \cdot$ corresponds to the subset ordering on the models of the formula ϕ. From Claim 6, a variable is dispensable iff it is False in all minimal models. In terms of the ordered solution domain, the variable has the value False in all the optimal solutions, i.e., False is settled in ϕ. □

6.5 Summary

This chapter shows how to extend a configurator with a function that helps the user to complete the configuration process whenever invoked. In the case of propositional configuration, this completion function is designed according to the shopping principle (Section 6.2). The motivation comes from an analogy with the shopping at a fruit stall: the customer is enumerating items he or she wants. This process terminates when the customer decides he or she has all

that was needed. Hence, the person who is shopping never needs to go through all the items in the stall and say for each item whether he or she wants it or not. In the case of dependencies between the items, this is not that easy and only certain items can be skipped and be not bought. The term dispensable variables (Definition 16) is introduced to formally identify the variables that can be eliminated (skipped) when the shopping principle function is invoked. The chapter shows how to calculate these variables using a SAT solver in Section 6.3. However, other ways for the calculation are opened by relating to propositional circumscription and Closed World Assumption in Section 6.2.4; for more information see Section 10.3 in the chapter on related work.

This chapter further generalizes the shopping principle to the case of non-propositional configuration relying on the notion of preference (Section 6.4.2). This generalization shows that the shopping principle can be seen as a preference for not-buying even though such presentation might appear counter-intuitive at first.

In summary, the contributions of this chapter are the following

- discussion and classification of possible scenarios of completion of configuration,
- formal definitions of the concept "making a choice between a set of user decisions",
- an algorithm that enables us to compute variables that can be eliminated without making a choice for the user, and
- a formal connection between the concept of choice and the concept of preference.

Chapter 7

Building a Configurator

While the previous chapters are concerned with the algorithms used in the configurator, this chapter describes a design of the configurator from a software engineering perspective. It should be noted up front that while the user interface itself poses some interesting challenges, this chapter is only concerned with the part responsible for the reasoning; this part is called the *reasoning backend* (or just *backend*). For more details on the user interface see [30, 29]. A configurator using a backend implemented by the author of this dissertation is called Lero S^2T^2 and is available online [174].

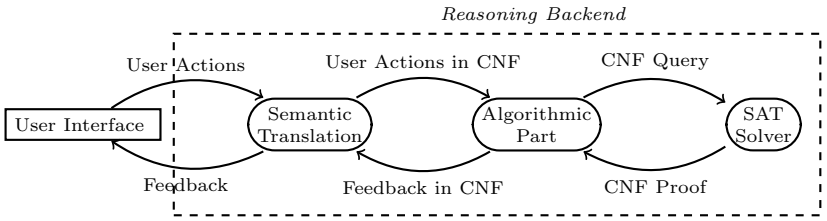


Figure 7.1: Data flow in the configurator

Figure 7.1 depicts the parts of the configurator and how they communicate. The user interface is responsible for loading the model and sending user actions to the reasoning backend. The algorithmic part represents the algorithm for finding bound literals as described by the previous chapters. The part between the user interface and the algorithmic part is called the *semantic translation* since it converts the feature model and user actions into a CNF representation according to the semantics of the modeling language. It is important to highlight that this translation takes place in both directions: the user sends decisions, the SAT solver sends proofs.

Even though Section 4.5 shows how to provide explanations in the form of resolution trees, the implementation treats explanations as sets. The reasons

for this are rather technical, the SAT solver used (SAT4J) supports proofs only in the form of sets of CNF formulas and the user interface does not support visualization of explanations as trees.

An important design decision in the implementation of the backend is its modularization. The modularization is guided by two main criteria: *separation of concerns* and *extensibility*. Separation of concerns led to the decision that the semantic translation is done in multiple steps, each step dealing with a conceptually different type of translation. Each of these steps is implemented as a separate component and thus can be easily replaced. Extensibility means that it should be easy to add optimizations such as the BIS-optimization described in Chapter 5.

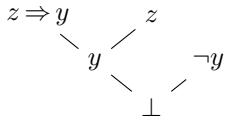
The architecture of the backend is described in the following section (Section 7.2). Section 7.3 describes the implementation of the backend, which follows the architecture described in the upcoming section. Section 7.4 discusses the advantages and disadvantages of the presented design.

7.1 Implementing Explanations

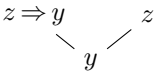
The implementation uses two different engines to compute explanations. One is built into the solver SAT4J, which is used as the underlying solver for the inference in the configurator. The second uses an external tool MUSER [140].

SAT4J uses the algorithm QuickXplain [111] to produce unsatisfiable cores of the given formula. Such core comprises leaves of some resolution tree that we need in order to provide explanations (see Section 4.5). To obtain such resolution tree, another SAT solver (called PidiSAT) is called on the core of clauses obtained from SAT4J. PidiSAT was developed by the author of this dissertation and has the ability to produce resolution trees for unsatisfiable inputs (unlike SAT4J). The reason why PidiSAT is not used in the inference algorithms is that it is far less efficient than SAT4J. However, the efficiency is not critical for constructing resolution trees because the solver is invoked on a significantly smaller set of clauses than the original problem. PidiSAT is available for download at Kind Software website [102]. The following example shows a typical scenario of how the tools are used.

Example 39. *Let the state formula be* $\phi \stackrel{\text{def}}{=} (x \vee u) \wedge (z \Rightarrow y) \wedge z$. *In this case* $\phi \models y$, *therefore* $\phi \wedge \neg y$ *is unsatisfiable. Invoking QuickXplain on* $\phi \wedge \neg y$ *yields* $(z \Rightarrow y) \wedge z \wedge y$ *as the clause* $x \vee u$ *is not necessary to show unsatisfiability. Subsequently, invoking PidiSAT on* $\phi \wedge \neg z$ *yields* $(z \Rightarrow y) \wedge z \wedge \neg y$ *the following resolution tree.*



Performing resolution skipping (see Figure 4.7) yields the following explanation.



In practice, the unsatisfiable core (the set of clauses necessary for showing unsatisfiability) is significantly smaller than the total number of clauses.

An important property of QuickXplain is that the returned core is subset-minimal. More precisely, if the algorithm returns the set of clauses C, then this set is unsatisfiable and any of its proper subsets $C' \subsetneq C$ is satisfiable. However, the algorithm does not not guarantee that there is no other set D that is also unsatisfiable but has fewer clauses than C.

The fact that QuickXplain guarantees subset-minimality is a welcome property from the user point of view because the explanations do not contain superfluous information. However, for larger instances the algorithm suffers from long response times. Hence, the tool MUSER [140] is used in such case. MUSER is based on the technique of iterative calls to a SAT solver [206, 142]. The tool produces an unsatisfiable subset of clauses as QuickXplain but does not guarantee subset-minimality. As the measurements show, this tool performs well even for large instances (Chapter 8). PidiSAT is used again to construct a resolution tree from the output of MUSER.

> ***Remark 7.31.*** *Since MUSER is written in C, a minor technical difficulty of integrating it with the Java implementation had to be overcome. The implementation outputs the formula into a file and subsequently invokes MUSER on that file. Since the formula may be large, the part corresponding to the semantics of the configured feature model is outputted only once per feature model. This common part is then concatenated with the formula representation of the current user decisions whenever needed.*

The implementation uses that tree data structure to represent the resolution trees (see Figure 7.2). In order to account for the different types used throughout the translation, the type of the tree's data is parameterized using the Java generics.

```
public class ResolutionNode<Constraint,Variable> {
        public final T clause;
        public final V variable;
        public final ResolutionNode<T,V> child_1;
        public final ResolutionNode<T,V> child_2;
}
```

Figure 7.2: A data structure for resolution trees.

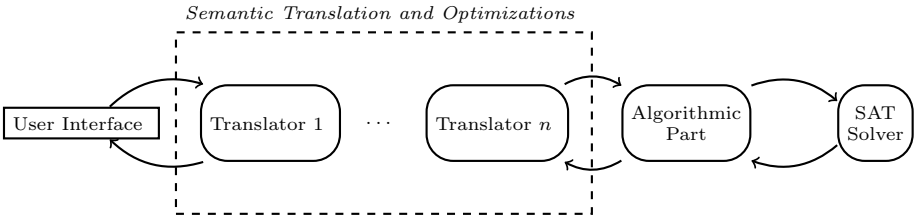


Figure 7.3: Architecture of the configurator

7.2 Architecture

Motivated by the objectives outlined above, the architecture of the configurator is a chain of components as depicted in Figure 7.3. The core algorithm (Chapter 4), the SAT solver, and the user interface are fixed. The middle part providing the semantic translation and optimizations is realized by a series of components called *translators*. The number of translators is not fixed which caters for the two principles mentioned above: the semantic translation can be done in a series of steps rather then in one step (separation of concerns), and different optimizations can be inserted or taken out of the chain (extensibility).

To enable the programmer to easily combine translators into different translator chains, the individual translators conform to some uniform conventions. These conventions apply to the form of the data being exchanged between the translators and to the interfaces the translators implement. We begin by discussing the form of the data being exchanged between translators.

7.2.1 Data Format

Recall that the reasoning backend starts with a feature model and translates it into a CNF representation (a set of clauses). When the reasoning backend produces explanations, it translates a set of clauses into some parts of the feature model and these are displayed to the user by the user interface.

The modeling language of feature models is quite heterogeneous: it is a tree of features accompanied by propositional formulas. However, this language has the nice property that the semantics of a future model is defined as a conjunct

of the semantics of well defined parts of the model (Section 2.4.1). We call these parts *modeling primitives*.

To achieve uniformity across formulas and feature models, we treat the models as flat structures—a set of modeling primitives. Table 7.1 lists the primitives that are used in Lero S^2T^2 to describe feature models along with logic formula corresponding to the semantics.

modeling primitive	**logic formula**
Root(r)	v_r
OptionalChild(c, p)	$v_c \Rightarrow v_p$
MandatoryChild(c, p)	$v_c \Leftrightarrow v_p$
Excludes(a, b)	$\neg(v_a \wedge v_b)$
Requires(a, b)	$v_a \Rightarrow v_b$
AlternativeGroup(p, $\{f_1, \ldots, f_n\}$)	$(v_{f_1} \vee \cdots \vee v_{f_n} \Leftrightarrow v_p) \wedge \bigwedge_{i<j} \neg(f_i \wedge f_j)$
OrGroup(p, $\{f_1, \ldots, f_n\}$)	$v_{f_1} \vee \cdots \vee v_{f_n} \Leftrightarrow v_f$
Constraint(ϕ)	ϕ
SelectedFeature(f)	v_f
EliminatedFeature(f)	$\neg v_f$

Table 7.1: Modeling primitives and their semantics

The names of the constructs closely follow the terminology used in the FODA notation (Section 2.4.2). The primitive Root(r) determines that the feature r is the root feature. The primitive OptionalChild(c, p) specifies that the feature c is an optional child of the feature p. Analogously, MandatoryChild(c, p) is used for mandatory children. The first argument to AlternativeGroup(p, $\{f_1, \ldots, f_n\}$) is a feature and the second is a set of features that form an alternative group of the parent feature. The primitive OrGroup is used analogously. The primitive Excludes(a, b) specifies that the features a and b are mutually exclusive and the primitive Requires(a, b) specifies that the feature a requires the feature b. The primitive Constraint(ϕ) is used for arbitrary propositional constraints.

The primitives SelectedFeature and EliminatedFeature stand out because they are not part of the original FODA notation. The main purpose of these primitives is to enable the user interface to specify user decisions by sending these primitives to the backend. Making user decisions a part of the modeling language has the following benefits. (1) It increases the homogeneity of the communication: there is only one language of communication for specifying both the initial feature model and for specifying user decisions. (2) It enables storing intermediate stages of the configuration process as a set of primitives, which is the same format as the format in which feature models are stored.

Decomposing the feature model into a set of primitives enables the reasoning backend to focus on one primitive at a time. More importantly, the choice of modeling primitives determines the granularity of explanations. Consider, for

instance, that the user interface sends the whole feature model as a sentence in some language, then it is not clear how to communicate to the user interface which parts of the model form the explanation. Whereas when the feature model is "chopped up" into a set of primitives, the backend can communicate explanations as subsets of primitives.

The more finely the modeling language is chopped up, the more fine will be the explanations. For instance, the primitive AlternativeGroup could be chopped up more finely by introducing separate primitive specifying a membership of a feature in a group and a primitive attaching a group to the parent feature.

> ***Remark 7.32.*** *If resolution trees were to be supported, the explanation would not consist merely of subsets of primitives but of graphs capturing the dependencies between the primitives.*

The concept of modeling primitives enables us to see both a feature model and a formula uniformly: a model is a set of primitives and a formula is a conjunct of its smaller parts. In particular, a CNF formula is a conjunct of clauses. In this chapter we use the term *constraint* to talk about both the modeling primitives and formulas. Analogously, features and logic variables are seen as instances of the same concept. We use the term *variable* to talk about either of these.

Hence, the principle that we follow when constructing translators is that each translator handles a certain type of constraints and variables. Intuitively, the first translator in the chain (Figure 7.3) handles the most high-level type of constraints while the last translator handles the most low-level type of constraints.

- *Any feature model and any formula is treated as a set of constraints. Additionally, user decisions are special types of constraints.*

7.2.2 Interface of Translators

```
interface ITranslator<Variable, Constraint> {
  void init(Set<Variable> vs, Set<Constraint> cs);
  void addConstraint(Constraint c);
  void removeConstraint(Constraint c);
  Set<Constraint> computeConstraints(Variable v);
  Set<Constraint> explain(Constraint c); }
```

Figure 7.4: The Java interface implemented by translators

This section concretizes the discussion by presenting the interface of the translators, which are chained together in order to provide the semantic translation in the backend. The description of the algorithmic part (Chapter 4)

already specifies the set of messages exchanged between the configurator and the user interface. This section discusses the same set of messages but at a more technical level and accommodates for the fact that the feature model must be translated into a CNF representation. Even though the current implementation operates on feature models with propositional semantics, the interface of the translators is designed in such way that the same interface can be used for more complicated semantics in the future.

All the translators implement the same generic interface in order to maximize uniformity over the translators. Additionally, the algorithmic part implements this interface as well. Hence, the last translator does not need to be treated specially since it connects to the algorithmic part as if it was another translator.

The motivation for this uniformity is that it enables each translator to rely on the interface of the following translator (or the algorithmic part), rather than on its concrete implementation. Consequently, this facilitates the a translator can be removed or inserted into the chain as long as it fulfills the interface.

Each translator is parametrized by the type of constraints and variables that it handles, e.g., the first translator handles feature modeling primitives and features. For better intuition, one may think of each translator as of a configurator for the pertaining type of constraints. Figure 7.4 presents a Java representation of the interface where Java generics are used to capture the parametrization. Let us discuss the individual methods.

The method init initializes a translator with a set of variables and constraints; all constraints must constrain only the variables given here. The method init is invoked just once—when the feature model is loaded. This method corresponds to the message INIT in Chapter 4.

> ***Remark 7.33.*** *The initialization phase fixes the set of considered variables. Nevertheless, it might be useful to have means for manipulating the set of variables on the fly.*

The purpose of the methods addConstraint and removeConstraint is to add and retract user decisions, respectively. The method removeConstraint must not be invoked on a constraint given via the init method. These methods generalize the messages ASSERT and RETRACT in Chapter 4. When the user selects or eliminates a feature, the user interface sends the appropriate primitive as a parameter of the method addConstraint of the first translator. However, the signature of the methods enables a translator to support more general decisions such as "feature x or y must be selected".

The function computeConstraints(Variable v) infers constraints for the given variable. This method generalizes the messages LOCK and UNLOCK in Chapter 4. If this method returns the empty set, the variable is not constrained,

i.e., the variable is unlocked. To signal that a variable is locked, the translator returns a set containing the appropriate constraint. For instance, in the case of feature models, to signal that a feature is locked is signaled by sending a set containing either the primitive SelectFeature or EliminateFeature. In the case of CNF, locking is achieved by sending the corresponding literal.

Clearly, this mechanism is more flexible than just the two messages LOCK and UNLOCK. If one would want to express, for example, that a certain number must be within a certain range, then it would suffice to extend the type of constraints with such constraint.

Note that in Chapter 4, the configurator infers constraints for all the variables after each user action. In this respect, the function computeConstraints corresponds to the body of the loop in TEST-VARS (Figure 4.3). Each of these iterations determines whether the inspected variable is bound or not. The reason why the interface provides the function computeConstraints separately is that it enables the user interface to inspect only certain variables. This can be used to improve the efficiency—if some portions of the model are not displayed, it is not necessary to query them.

The function explain(Constraint c) provides an explanation for a given constraint c. The constraint c must have been previously returned by computeConstraints. The method must guarantee that the returned explanation contains only such constraints that were added via init or addConstraint and not removed in the meantime. This method corresponds to the message EXPLAIN in Chapter 4.

Because the translators are directly implemented in Java, it is hard to capture in the interface that the translator is expected to be connected to the translator that follows in the translator-chain (Figure 7.3). The convention used in the implementation is that the following translator is specified in the constructor of each translator. Hence, each translator has a constructor of the form Translator(ITranslator<CT,VT> followingTranslator).

A specific chain of translators is then constructed by nested constructor calls where the algorithmic part is constructed first, i.e., the chain of translators T1,...,TN followed by the algorithmic part A is constructed by the Java code: new T1(new T2(... new TN(new A())...)).

> ***Remark 7.34.*** *The convention just outlined could be resolved more elegantly if the configurator was implemented in the context of some middleware that would enable modeling the conventions explicitly by a software architecture description language [146].*

- *All translators implement an instantiation of the same generic interface. This enables constructing different translator chains which are checked by the Java type system.*

7.3 Implementation

This section describes the implementation of the backend in Lero S^2T^2. It is not the objective of this section to describe the actual Java code but to highlight the main characteristics of the implementation. The description begins with the different types of constraints used in the backend.

7.3.1 Constraint Types

Each translator implements the interface discussed above (see Figure 7.4) and translates between two types of constraints: certain type of constraints are sent to the translator through the methods of the interface and certain type of constraints is obtained from the translator that follows in the chain.

Constraint types are represented as is common in object oriented programming. Each constraint type corresponds to a Java class whose member fields determine the pertaining variables. To represent a family of constraint types, the classes inherit from a class representing that particular family. For instance, the primitive SelectFeature is represented as a class with a member field specifying the selected feature. Since SelectFeature belongs into the family of feature model primitives, it inherits from a class representing the family of feature model primitives (in the following text this class is denoted as `FMP`).

The implemented families of constraints are listed in Table 7.2. The table contains a brief description of each of these families and a description of the constraint and the variable type that are used to instantiate the ITranslator interface (Figure 7.4).

Name	Description	Constraint Type	Variable Type
`FMP`	feature models	feature model primitive	feature
`EPL`	extended propositional logic	extended formula	variable
`PL`	propositional logic	formula	variable
`CNF`	conjunctive normal form	clause	variable

Table 7.2: Constraint families

The family `FMP` contains the feature primitives presented above (see Table 7.1). The family `EPL` contains standard logic connectives but additionally

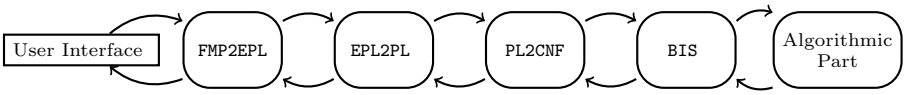


Figure 7.5: Concrete architecture of the configurator

contains the construct `select(p, v1, ..., vN)` specifying that exactly one of the `v1, ..., vN` must be TRUE iff `p` is TRUE. This construct is used in the translation of the semantics of alternative groups. The reason to have a special construct for this is that translating it into standard logic connectives can be done in multiple ways[11, 184, 62, 137]. The family `PL` is a standard propositional logic without the construct `select`. Finally, the family `CNF` is used to capture a formula in the conjunctive normal form.

The task of the translators is to provide a gradual translation from the family `FMP` to the family `CNF`. This is discussed in the following section.

7.3.2 Implementation of Translators

Figure 7.5 shows the translators that provide the semantic translation in Lero S^2T^2. The translator `FMP2EPL` translates from feature primitives to propositional logic while using the construct `select`. The translator `EPL2PL` desugars this construct leaving only the standard connectives. The translator `PL2CNF` translates propositional logic into the conjunctive normal form; the current implementation is using de Morgan's laws and therefore does not add any new variables (see Section 2.1.3).

These translators enable us to construct a chain from feature model primitives to conjunctive normal form. Additionally, there is the translator `BIS` implementing the BIS-optimization (see Chapter 5). Recall that the optimization expects conjunctive normal form and produces conjunctive normal form. Therefore, this translator handles the constraint family `CNF` and is connected just after the translator `PL2CNF`. Thanks to the modularization principles used in the design, the translator `BIS` can be taken out of the chain so that the translator `PL2CNF` is connected directly to the algorithmic part.

> ***Remark 7.35.*** *Observe that the translator `BIS` expects the same type of constraint from the translator on its left and from the translator on its right. This is in fact a necessary condition for a translator to be freely removable from the chain.*

All of the translators work in a similar fashion. Each of the translators maintains a mapping between the two pertaining constraint types and a mapping between the two types of variables. These mappings are initialized in the method init and are updated in the method addConstraint. The other methods of the

interface use these mappings to convert between the two types.

As an example, let us look on the implementation of the translator `FMP2EPL`— from feature model primitives to extended propositional logic. This translator implements the interface `ITranslator` with the family `FMP`. The constructor of the translator expects a reference to the translator that follows in the chain and implements the interface `ITranslator` instantiated with the constraint family `EPL`. The translator `FMP2EPL` maintains a one-to-many mapping from feature model primitives to formulas and a one-to-one mapping from features to variables.

The method init populates these mappings with the initial set of primitives and the corresponding semantics expressed as the extended propositional logic formulas.

The method addConstraint(FeatureModelPrimitive p) expects one of the primitives SelectFeature and EliminateFeature; it translates the primitive into the corresponding literal l and adds the correspondence between the primitive and the literal into the mapping. Finally, it calls the method addConstraint of the following translator with l as the argument.

The method `removeConstraint(FeatureModelPrimitive p)` removes the primitive `p` from the mapping and calls the method `removeConstraint` in the following translator for all those extended propositional logic formulas that are not mapped to. The reason why the method cannot simply remove all the formulas that correspond to the semantics of the primitive `p` is that the mapping is not necessarily injective.

The method `computeConstraints(Feature f)` looks up in the variable mapping the variable corresponding to the feature `f` and calls the method `computeConstraints(Variable v)` of the following translator. This method either returns the empty set (if the variable is unbound) or a set containing the bound literal. If the following translator returns the empty set, then the methods returns the empty set since the feature is not bound. If the following translator returns a bound literal, the method converts the literal to the appropriate primitive and returns a singleton set containing that primitive.

The implementation of the method explain(FeatureModelPrimitive p) first calls the method explain of the following translator for each formula corresponding to the semantics of the primitive p. Then it computes the union of these formulas and converts them back to a set of primitives using the inverse of the mapping between primitives and formulas.

The other translators are implemented analogously. However, there are some differences in terms of the cardinalities of the mappings. For instance, the translator `BIS` maintains a many-to-one mapping between variables since variables in the same BIS are represented by a single variable (see Chapter 5) while the

other translators maintain a one-to-one mapping between variables. For more details see the source code[1].

7.4 Summary of the Design

This chapter has presented the design of the reasoning backend used in the configurator Lero S^2T^2. The architecture of the backend resembles the well-known architecture *pipe-and-filter*. However, compared to the traditional setting, the components in the chain communicate in both directions since the backend needs to provide explanations.

All the information exchanged between the components is represented as sets of constraints. The type of these constraints depends on the phase of the semantic translation. As is common, the translation starts from the most high-level type of constraints and proceeds to the most low-level type of constraints. In some sense, this follows the well-known pattern of step-wise refinement [200, 57, 18, 17].

If the feature model is to be treated as a set of constraints, it needs to be chopped up into a set of primitives. Even that is not all that novel. Indeed, the primitives used to describe feature models correspond to relations in the entity-relation modeling, such as UML. Hence, it is possible to assume that this approach to capturing feature models could be generalized to other models as well.

The components comprising the implementation are components in the object oriented sense, i.e., the implementation of the translators are separate classes that communicate with each other through an interface. This interface is prescribed by a generic interface which is instantiated by the specific type of constraints that the given translator handles. An alternative to this setup would be cross-cutting components used in feature oriented or aspect oriented programming. In such setup there would be a single class providing the semantic translation. This class would be modified by different features (or aspects) providing the functionality of different translation phases.

The disadvantage of components organized in the object oriented sense is efficiency. Each of the translators needs to maintain a mapping between the two pertaining types of constraints. Whenever a translator responds to some user action, it needs to perform a look-up in these mappings. The advantage of this approach is that the programmer that is adding new translators, or is composing some existing ones, can treat the translators as black-boxes. If the translation where organized in cross-cutting components, this additional book-

[1] http://kind.ucd.ie/products/opensource/Config/releases/

keeping could be eliminated. Nevertheless, at the price of making the code more entangled and hampering extensibility. However, the look-up routines did not come up as problematic during the profiling of the implementation.

This particular design has proven to be useful throughout the development of Lero S^2T^2. According to our experience, the activities that benefit the most from this design are experimentation and evolution as a new type of functionality can be implemented as a separate translator and this translator is inserted into the chain only by the programmer who is testing it.

In summary, the contributions of this chapter are: a description how to implement a configurator in an object oriented language and a technological solution for providing explanations when using a SAT solver that does not return resolution trees.

Chapter 8

Empirical Evaluation

This chapter presents a number of measurements performed using the described implementation of the configurator run on third party instances. The motivation for these measurements is twofold. Firstly, the measurements should show whether the described algorithms are suitable for the interactive setting or not. In particular, whether the response time is acceptable for a human. Secondly, the measurements should show the impact of the proposed optimizations.

Section 8.1 describes the testing environment and process. Section 8.2 describes the test data and their main characteristics. Section 8.3 presents measurements that were done for the response time of the algorithm computing bound variables during an interactive configuration process. This response time determines how long the user has to wait before getting a response from the configurator after he or she made a decision. The effect of the BIS-optimization (Chapter 5) is also studied. Section 8.4 studies how many times the SAT solver was invoked when computing bound variables. Section 8.5 studies the explanation sizes and Section 8.6 studies the explanation response times. Section 8.7 presents measurements that were done for the technique of computing dispensable variables. Two aspects of this technique are studied: the response time and the effectiveness. Section 8.8 evaluates whether the number of measurements gives us a good estimate of the overall behavior of the configurator using the method of confidence intervals. Section 8.9 compares different types of explanations and argues that explanations based on resolution trees (Section 4.5) are more informative than explanations highlighting only pertaining user decisions. Finally, Section 8.10 draws conclusions from the measurement and summarizes the results.

The measurements in this chapter are presented in the form of *histograms*, i.e., the horizontal axis corresponds to the value being observed and the vertical axis is the number of times that particular value occurred in the set of

observations in questions (*frequency*).

8.1 Testing Environment and Process

The measurements were performed on nine machines where each has Quad-Core AMD Opteron^{tm} 64-bit processors with 512 kB cache. Eight out of these nine machines have 8 processors while one of these machines has 16 processors. However, since no multithreading was used this does not affect the performance. The installation of the Java virtual machine (Java HotSpot^{tm} 64-Bit Server VM) was identical on all the machines and its heap size was limited to 2 GB for each measurement.

All the measurements were performed in the following three phases.

- *Test generation*—in this phase testing data were generated and recorded in a dedicated format. In particular, the testing data were sequences of user decisions recorded as a file whose each line recorded which user decision was made (selection or elimination of a variable).
- *Test execution*—in this phase the data generated in the previous phase were read in and executed while recording the data to be collected (response times, etc.). It should be noted that all the times are *stopwatch times*, i.e., obtained by subtracting the time observed before the measured operation from the time observed after the operation.
- *Data aggregation*—using custom scripts and small Java programs the times were aggregated to obtain average values and data for the plots. The plots were rendered using the program gnuplot [82].

8.2 Test Data

Seven instances were chosen to perform the evaluation. Three of these instances are large feature models with up to hundreds of features, available at the feature model repository [66]: E-shop, Violet, and Berkeley. A toy example representing a feature model of a T-shirt was added as this example commonly occurs in articles on configuration. The largest instance evaluated is the Linux kernel variability model [22].

Two instances are not feature models but randomly-chosen DIMACS format (Center for Discrete Mathematics and Theoretical Computer Science format) examples used in SAT competitions and benchmarking: 3blocks and rocket. In the following text we refer to the feature models and the Linux kernel instance

Name	#variables	#clauses
E-shop	287	420
Violet	170	341
Berkeley	94	183
T-shirt	16	40
3blocks	283	9690
rocket	351	2398
Linux kernel	8092	44533

Table 8.1: Instances used for configuration and their properties

Name	Length	#decisions	noop	BIS	Cat	Ord	BCO
E-shop	144.3	7215	53	42	53	9	7
Violet	50.34	2517	16	15	16	4	4
Berkeley	25.12	1256	14	11	14	7	5
T-shirt	4.7	235	8	7	8	5	5
3blocks	6.7	335	655	616	557	3418	2671
rocket	7.52	376	166	163	158	274	260

Table 8.2: Overview of the measurements with times in ms

as *variability models* and the instances from the SAT competition as *SAT competition instances*.

Table 8.1 lists the sizes of the instances; for feature models it lists the size of the formula capturing the model's semantics.

Sequences of user decisions were generated using a pseudo-random generator. Each sequence is making decisions until all variables are bound, i.e., until the process is complete. Fifty different sequences were generated for each instance. Note that different user decision sequences may have different lengths even for the same instance (Section 8.8 argues that the choice of 50 sequences is sufficient).

8.3 Configurator Response Time

The objective is to collect information about the response time of the configurator for different optimizations. The optimizations considered are the following.

- **BIS**: the BIS-optimization collapsing literals in bi-implied sets (Chapter 5).
- **Cat**: the adding of bound literals to the formula. The motivation for adding bound literals is to give more information to the solver, i.e., these bound literals should serve as **cat**alysts (Figure 4.5).
- **Ord**: the optimization modifying how the SAT solver looks for solutions.

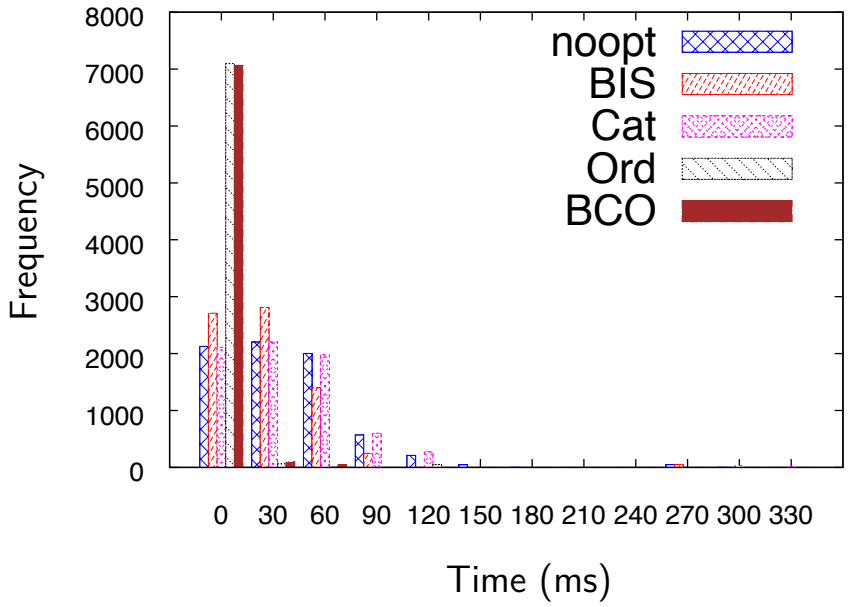


Figure 8.1: Distribution of response times for E-shop

In particular this optimization changes the **ord**ering of the search space traversal (Section 4.4).

The measurements were carried out with five different combinations of the optimizations. One version, denoted as **noopt**, does not use any of these optimizations. Another three versions use one of these optimizations. And, the last version, denoted as **BCO**, uses all of them.

For each optimization combination, instance, and user decision sequence the measurement was repeated ten times and averaged to filter out noise caused by the operating system and the machine[1].

Table 8.2 summarizes the results of the measurements. The first column is the name of the investigated instance. The second column is the average length of a user decision sequence; observe that this is in fact 50 times the first column since 50 sequences were evaluated for each instance. The third column is the total number of user decisions over all sequences. Each of the last five columns shows the average over all user decisions and sequences for one of the version of optimizations in milliseconds.

Figures 8.1–8.6 present the different time distributions for different instances and optimization combinations, except for the Linux kernel instance. The horizontal axis is the response time and vertical corresponds to the number of

[1]The solver has a random component but this is initialized by a constant seed and therefore is not observable in repeated measurements.

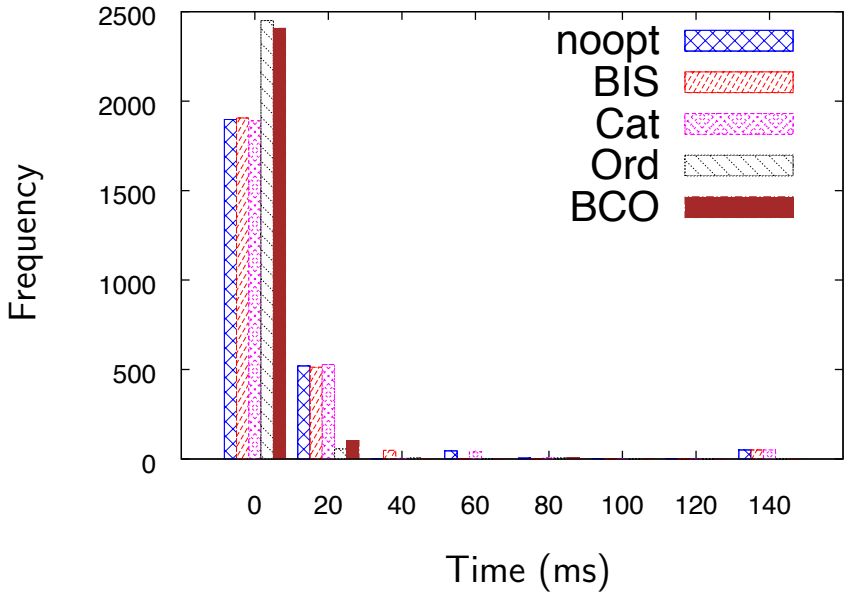


Figure 8.2: Distribution of response times for Violet

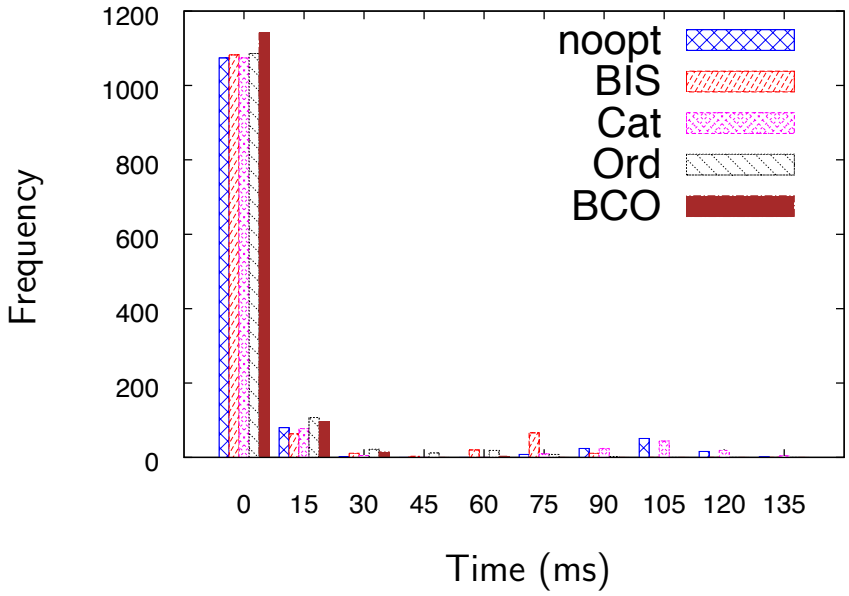


Figure 8.3: Distribution of response times for Berkeley

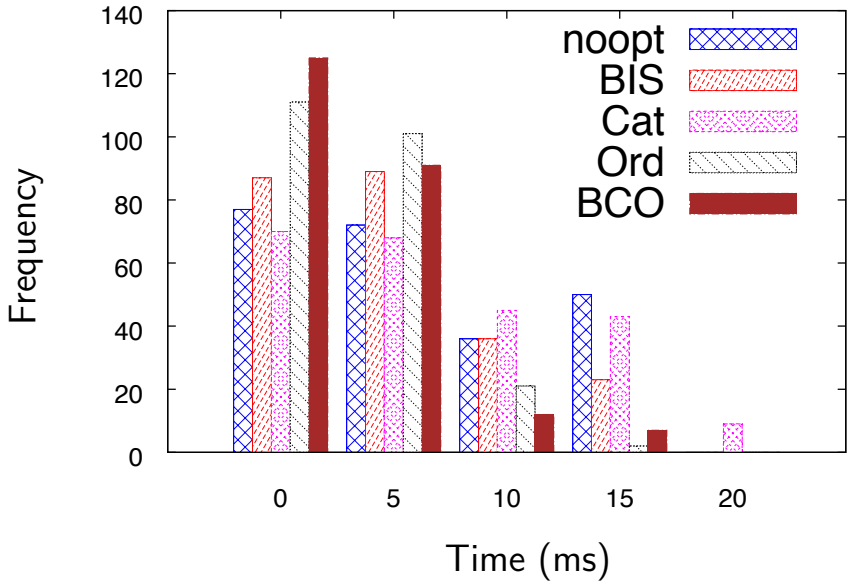


Figure 8.4: Distribution of response times for T-shirt

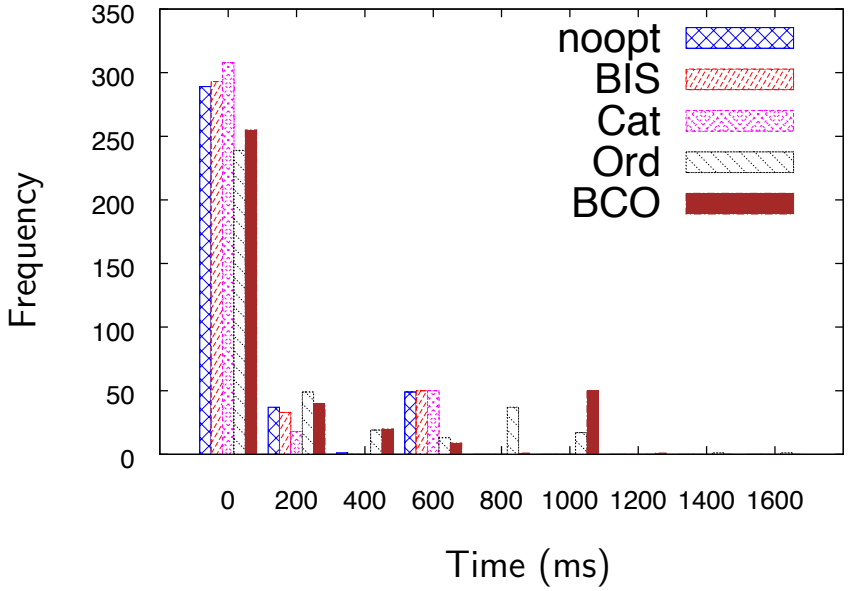


Figure 8.5: Distribution of response times for rocket

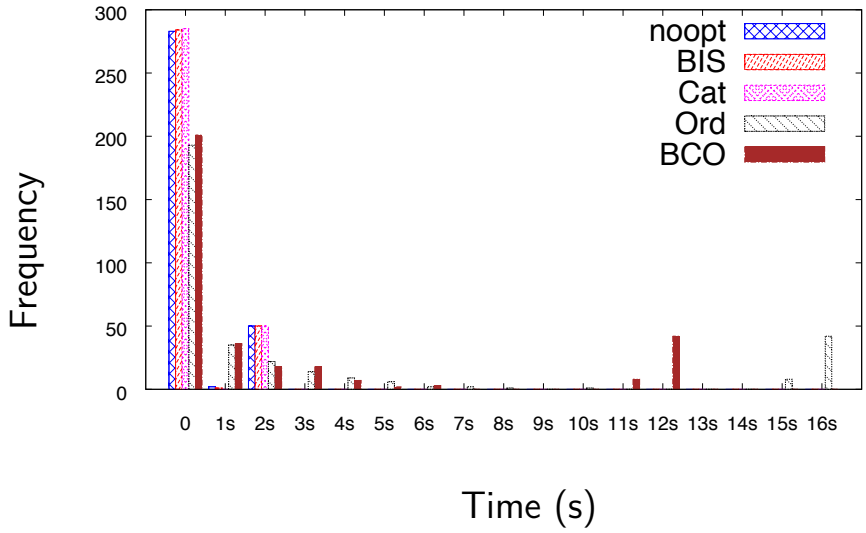


Figure 8.6: Distribution of response times for 3blocks

occurrences of the response time. All the times are in milliseconds except for the diagram concerning the instances 3blocks. The response times are clustered to obtain a histogram. For instance, in the E-shop distribution there are over 7000 occurrences of response times below 30 ms for the BCO optimization combination. Since 7215 cases were evaluated for the E-shop instance, over 97% of these cases were below 30 ms using the BCO optimization combination.

Linux Kernel Instance Since the Linux kernel instance is substantially larger than the other instances, only one iteration for each user decision sequence was performed as otherwise the measurements would require a couple of months. However, Section 8.8 shows that the variations over different iterations and sequences are small, which suggests that the performed measurements provide a good estimate of the overall behavior.

Another issue was with evaluating the instance *without* the **ord** heuristic. The response times were significantly worse without the heuristic—around 1 minute per user decision. Hence performing a precise evaluation without the heuristic is not possible within a practical time frame.

The Linux kernel instance yielded the following observations. The average length of a user decision sequence was 4195.28, i.e., the total number of observed cases was 209764.

The evaluated optimization combinations are the following **O**—**ord** optimization, **BO**—**ord** optimization coupled with the BIS-optimization, and **BrO**—**ord** optimization coupled with the BIS-optimization that is *not* using

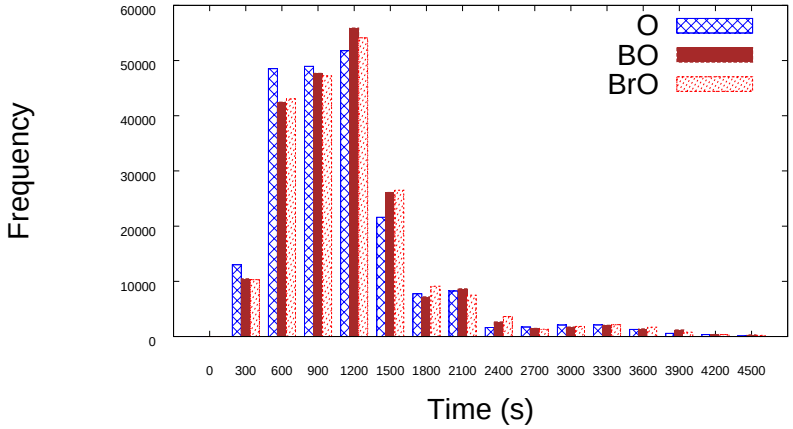


Figure 8.7: Distributions of response times for the Linux kernel benchmark with the tail cut off

the heuristic for picking a representative (see Chapter 5).

The following table presents the average time in milliseconds observed for the different optimization combinations.

	O	**BO**	**BrO**
average time [ms]	1268	1313	1310

Figure 8.7 presents the distribution of the response times where the tail of the distribution was cut off for clarity. The tail encompasses 16, 229, and 126 cases for the settings **O**, **BO**, and **BrO**, respectively. Out of the 209764 cases these all form less than 0.02%. The longest response time observed was 10 seconds, which occurred just once. All the other times in the tail were below 6 seconds.

8.3.1 Discussion

Let us look more closely at the different distributions obtained from the measurements. First, let us look on the feature model instances E-shop, Berkeley, Violet, and T-shirt.

Out of the optimizations **Ord**, **BIS**, and **Cat**, the **Ord** optimization is a clear winner: the total average is the lowest and the majority of response times are within 20 ms for all the feature model instances for this optimization. The combination of all the optimizations (**BCO**) performs only slightly better than the optimization **Ord** on its own.

The second comes the BIS-optimization: in the largest instance (E-shop) the average time was reduced by 20% and the maximum times are lower than for the versions **Cat** and **noopt**. Moreover, the number of response times that are greater than 60 ms is low.

The optimization **Cat** does not seem to be very helpful. For some instances it even adds a tail to the distribution.

Now let us look at the Linux kernel instance. Clearly, the **Ord** was greatly important since it wasn't possible to evaluate the instance without it. The BIS-optimization did not prove to be useful for the instance. In fact, the optimization slightly worsens the response times. However, this worsening changes the average response time by only a few milliseconds. This is most likely due to the fact that there are few BISs (strongly connected components in the implication graph) in the instance.

Most of the response times are around 1 second (see Figure 8.7). The times around 2 seconds appear in ca. thousand cases, which is 0.5% of the total number of cases. The distribution has a very long tail, meaning that the user will have to wait up to 10 seconds in some instances. However, these cases are very rare. In particular, the response time of 10 seconds occurred just once.

The numbers are very different for the instances coming from SAT competitions (3blocks and rocket). The configurator performs much worse with the optimization **Ord** than without it. That is already obvious from the averages (the average increased six times for 3blocks). The explanation for this phenomenon is that these instances are more difficult for the SAT solver and the **Ord** heuristics takes away some smartness from the solver. In the case of the feature model instances taking away the smartness makes up for the fact that the satisfiability tests find more quickly the literals that are not bound.

The **BIS** optimization is only moderately successful on these instances. However, the combination of the optimizations mitigates the worsening in times caused by the **Ord** optimization.

A difference between the feature model instances and the SAT instances can also be seen in Table 8.2. Both of the SAT instances have more variables than the feature model instances but a configuration process on the SAT instances is shorter. This indicates tighter dependencies between the variables: a small set of decisions determines values for all the other variables.

An interesting phenomenon is that all the distributions have long tails, i.e., there are many short times. This is explained by the fact that as a configuration process progresses, the instance is getting more constrained by the user decisions made so far. Effectively, the satisfiability tests are becoming easier. The fact that there are so many short times, signifies that these tests become fast only after a few decisions.

> ***Remark 8.36.*** *To analyzed in more detail the bad times, one could analyze only some prefixes of the user decision sequences. However, it is not clear how this cut off point should be chosen.*

- *The optimization **Ord** is highly efficient for loosely constraint instances. the optimization **BIS** yields only moderate speedups but does so for all the instances.*

8.4 Number of Calls to The Solver

In Chapter 4 we show that the solver is called at most $n+1$ times after each user decision, where n is the number of variables (Claim 3). However, we hypothesize that the number of calls is in practice significantly lower (see also Remark 4.13). This section presents the number of calls to the solver observed in the evaluated benchmarks. Even though $n+1$ is the maximal number off calls, for the purpose of the following calculations we assume it is n since the SAT solver is called once when the configurator is initialized in order to test that the given instance has at least one valid configuration (the theoretical part considers this to be a precondition).

Even though the number of variables is constant throughout the configuration process, the instance becomes easier throughout the process as the user is making decisions. Hence, the number of calls the solver is interesting *with respect to the number of variables whose values are not determined by a user decision.*

Figures 8.8–8.13 show distributions of percentages calculated as $100 * c/f$, where f is the number of variables whose values are not determined by user decisions and c is the number of calls the solver. These measurements were performed over the same set of sequences of user decisions as the measurements for response times.

The Linux kernel benchmark is not plotted as the numbers of calls to the solver are predominantly smaller than the number of variables not decided by the user. This is due to the fact that the instance was evaluated only with the optimizations containing the **ord** heuristic. Out of the 209764 cases, the solver was called less than 10 times in 90% cases. The maximum number of calls the solver observed was 725, which occurred only once.

8.4.1 Discussion

As expected, the optimization **Ord** dramatically decreases the number of calls to the solver. With this optimization, most of the times the solver needs to be called only 10%-times with respect to the number of variables not decided by the user. In the Linux kernel instance the number of calls to the solver was almost constant ranging between 0–10 calls the solver in 90% of the cases. A few cases occurred where the solver needed to be called a few hundred times. help these cases, however, where extremely rare.

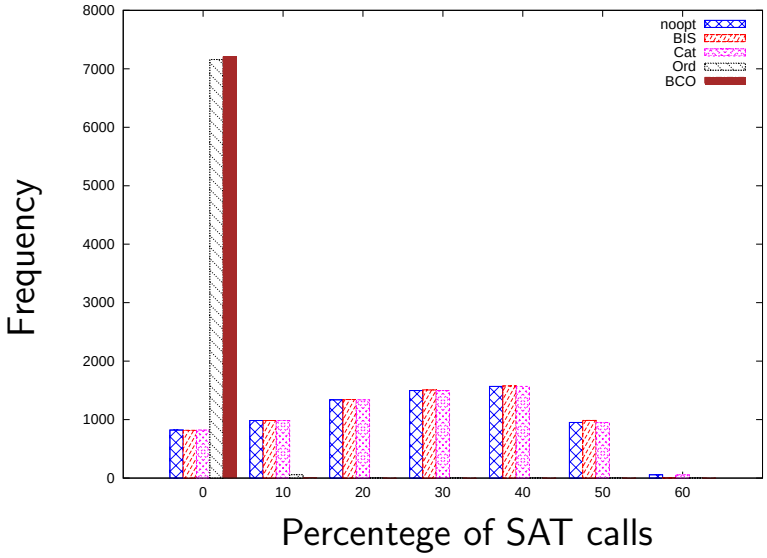


Figure 8.8: Percentages of number of calls to the solver for E-shop

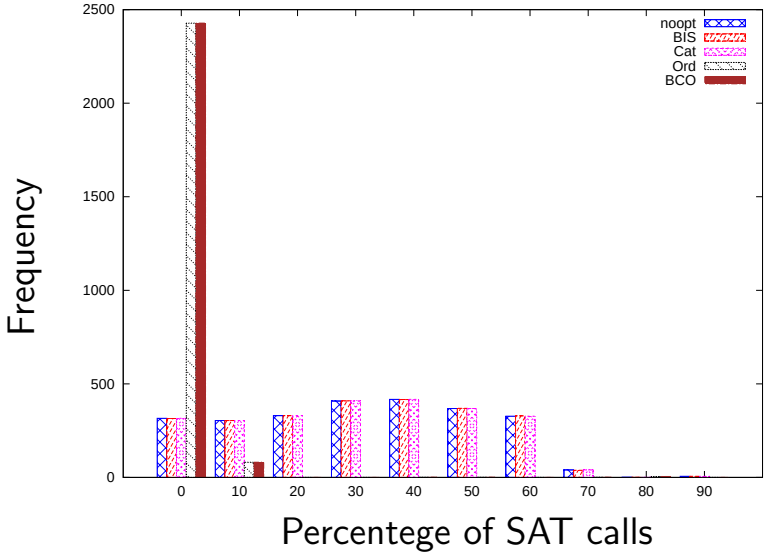


Figure 8.9: Percentages of number of calls to the solver for Violet

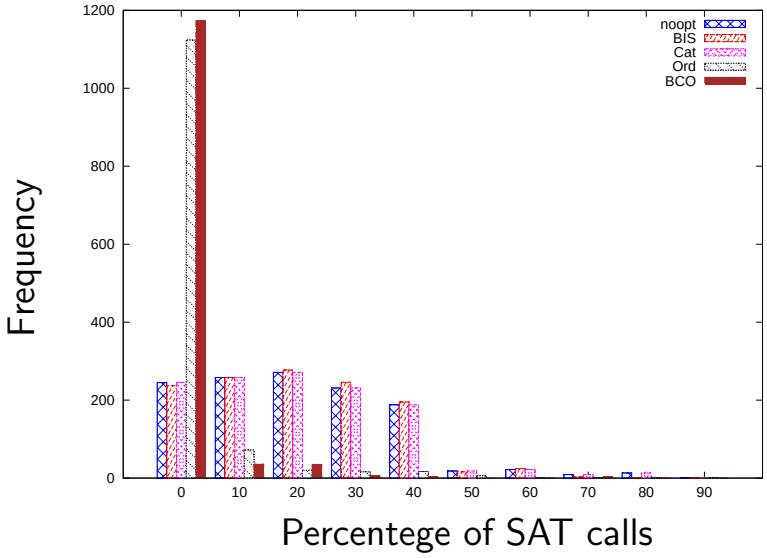


Figure 8.10: Percentages of number of calls to the solver for Berkeley

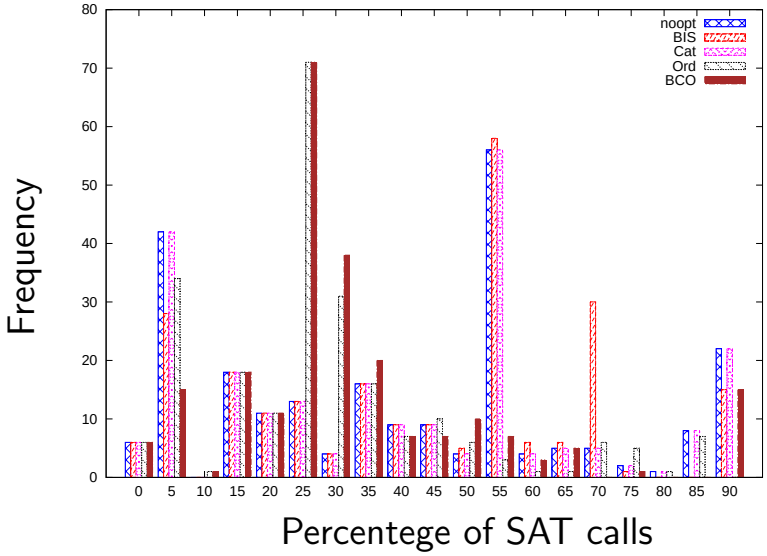


Figure 8.11: Percentages of number of calls to the solver for T-shirt

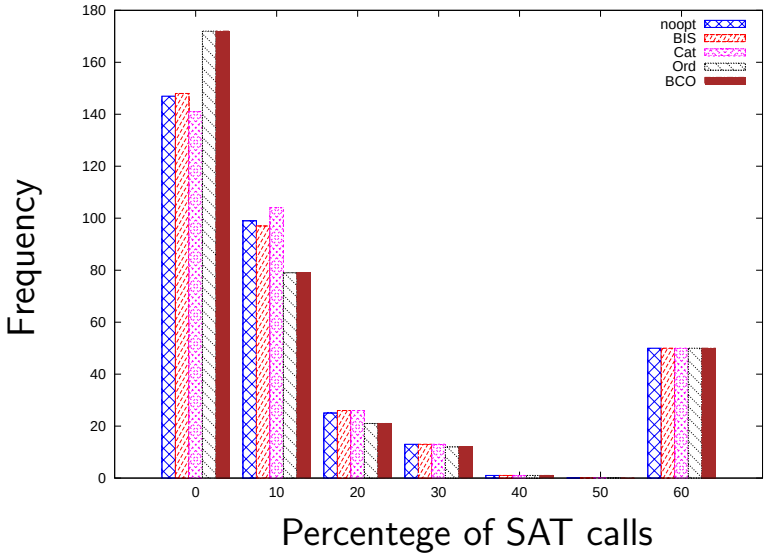


Figure 8.12: Percentages of number of calls to the solver for 3blocks

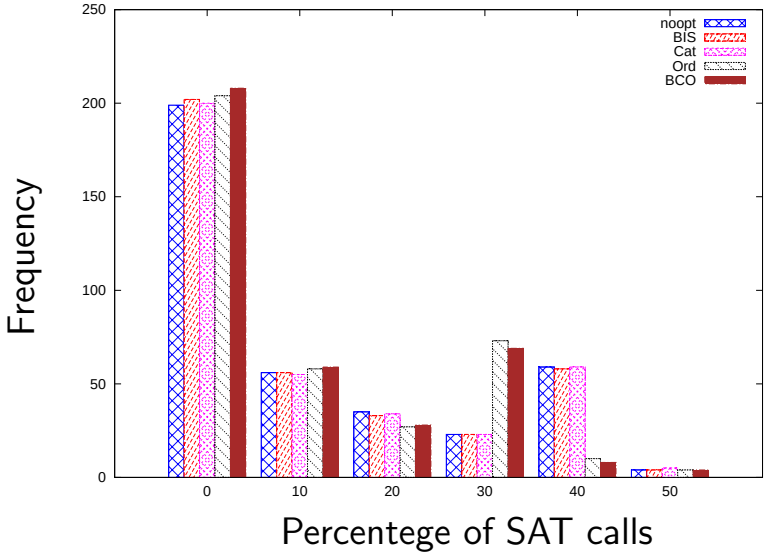


Figure 8.13: Percentages of number of calls to the solver for rocket

The distributions for the other optimizations (**noop**, **BIS**, and **Cat**) exhibit only minor differences from one another, with the exception of the T-shirt instance. Interestingly, the structure of these distributions varies significantly across the benchmarks. In the E-shop instance the distributions peak at 40% and quickly drops to 0 at 60%. In the Violet instance, the distributions slowly ascend to the peak at 40% and drop to 0 at 70%. In the Berkeley instance, the distributions peak at 20% and drop to 0 at 50%. The distributions of the T-shirt instance are somewhat erratic, which most likely is due to the small size of the instance.

Both of the SAT competition instances yield similar distributions. Each peaks 10% and descends quickly to 0. An interesting phenomenon appears in the 3blocks instance, where ca. 50 cases needed 60% invocations of the solver for all the optimizations. This is most likely due to situations where many variables are bound and therefore the **Ord** heuristic does not have a great effect.

In summary, the **Ord** heuristic decreases the number of calls the solver significantly. If the heuristic is not used, the expected number of calls to the solver is in the range of 20%–40% of the number of variables not decided by the user.

8.5 Explanation Sizes

In this section we overview of the evaluation of the explanation sizes. Two different mechanisms were used to produce explanations. One mechanism was QuickXplain [111], which guarantees subset-minimality but does not scale well. The second mechanism was the tool MUSER [140]. Both of these mechanisms were used with the combination of an additional SAT solver for constructing resolution trees from unsatisfiable cores (for details on how these tools are integrated into the explanation mechanism see Section 7.1).

MUSER was used for larger instances (SAT competition and Linux kernel). The rest of the instances were evaluated with QuickXplain. It was not possible to evaluate the 3blocks benchmark because the observed proofs were too large (over 2^{31}), which caused unwieldy response times.

Figure 8.14 summarizes which techniques were used for the individual instances.

What is in particular interest for us is the effect of the BIS-optimization on explanations' size. Recall that the BIS-optimization partitions the set of literals into sets of equivalent literals in order to reduce the number of queries to the solver. Each set of equivalent literals is represented by one literal—the representative.

Name	Explanation mechanism
E-shop	QuickXplain
Violet	QuickXplain
Berkeley	QuickXplain
T-shirt	QuickXplain
Linux-kernel	MUSER
rocket	MUSER
3blocks	not evaluated

Figure 8.14: Techniques used for explanations (see Section 7.1 for implementation details)

We show how to reconstruct resolution trees obtained on the optimized formula and we show that the choice of representatives influences the size of the reconstructed proof. Further, we devise a heuristic that picks such representatives that are likely to yield smaller reconstructed proofs (Section 5.2).

To evaluate the effectiveness of this heuristic, the reconstruction was once done with a randomly picked representatives and once with representatives picked according to this heuristic.

Hence, the following settings were evaluated: no optimization (**noop**), BIS-optimization (**BIS**), and BIS-optimization with random representative (**BIS-RR**). The following table presents the average sizes of the observed explanations.

Name	noop	BIS	BIS-RR
E-shop	8	7	15
Violet	4	4	4
Berkeley	8	10	15
T-shirt	5	6	10
Linux-kernel	15	16	16
rocket	6,001,001	3,519,066	6,153,248
3blocks	n/a	n/a	n/a

Figures 8.15—8.19 depict the distributions of explanation sizes for the individual instances. The distribution for the rocket instance is not plotted due to its unusual structure. Out of the 349 measured cases 124 cases yield explanations of size smaller than 100. The rest of the cases are evenly split up to the maximum size, where the maximum size was $4.6 * 10^8$ for **noop**, $1.6 * 10^8$ for **BIS**, and $1.7 * 10^9$ for **BIS-RR**.

8.5.1 Discussion

The explanation sizes for the variability models are small. Even for the Linux kernel instance the average size is 16, which is well-manageable for a user. How-

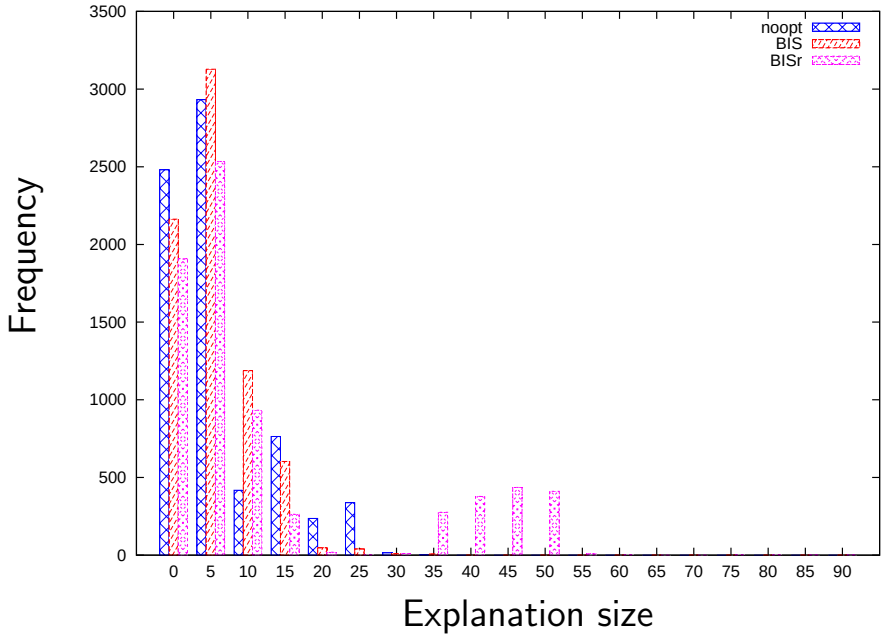


Figure 8.15: Distribution of explanation sizes for E-shop

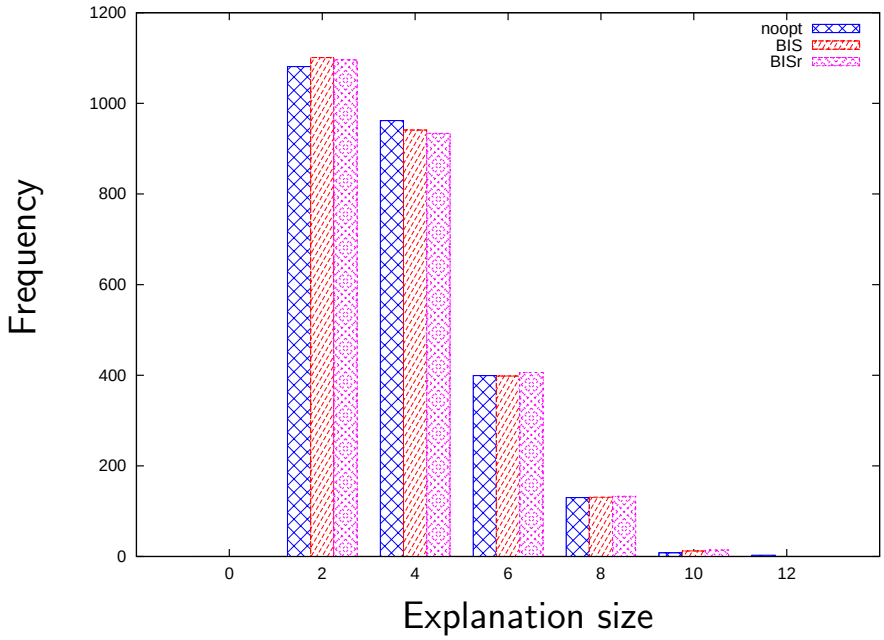


Figure 8.16: Distribution of explanation sizes for Violet

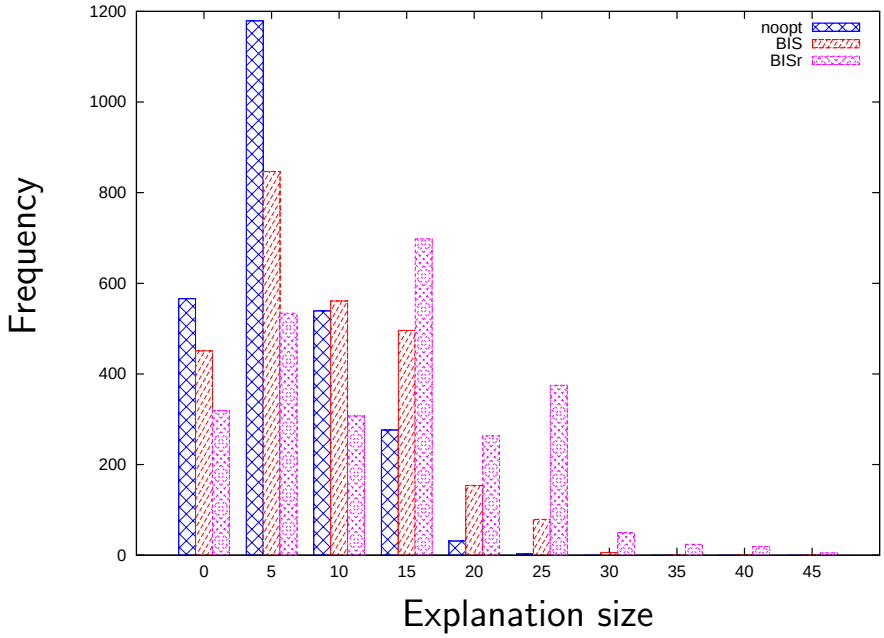


Figure 8.17: Distribution of explanation sizes for Berkeley

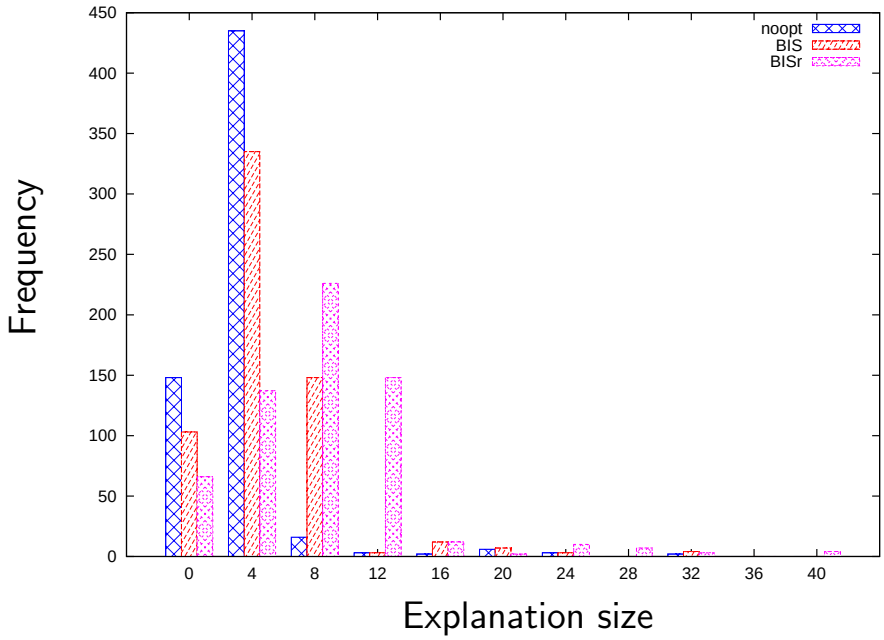


Figure 8.18: Distribution of explanation sizes for T-shirt

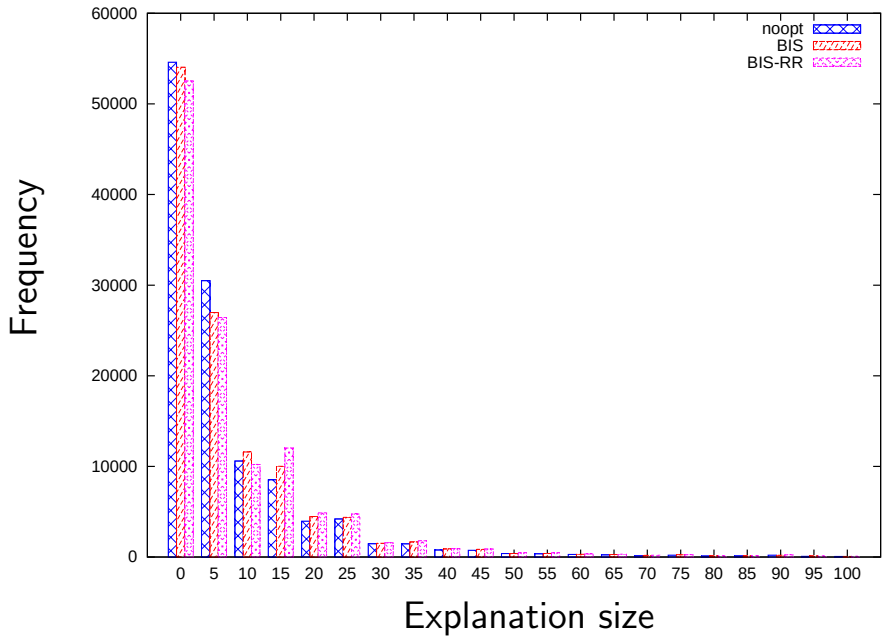


Figure 8.19: Distribution of explanation sizes for Linux kernel

ever, for the instances from the SAT competition the obtained explanations turned out to be unwieldy.

The heuristic used in BIS-optimization for choosing representatives proven to be important. For instance, in the E-shop instance not using the heuristic doubles the average explanation size and significantly prolongs the tail of the distribution.

Now let us look at the comparison of sizes of explanations constructed *without* the BIS-optimization and the sizes constructed in the presence of the heuristic. As expected, in most cases the reconstruction of the proof yields bigger proofs than those obtained without the optimization. However, this increase in size is relatively small. The explanation average grew by 1 in Linux kernel and T-shirt, it grew by 2 for Berkeley, and it didn't change for Violet.

Interestingly, the reconstructed explanations' sizes are smaller for the E-shop and rocket instances. Most likely, the reason for that is that the algorithm computing the unsatisfiable core (QuickXplain or MUSER) is operating on a smaller formula and therefore is capable of finding smaller cores.

In summary, the explanation sizes are small for variability models but tend to grow rapidly for complex formulas. The proposed heuristic for choosing a representative in the BIS-optimization is important for the explanations' size, and, in some cases the reconstructed proof can be smaller than a proof obtained from the unoptimized formula (E-shop and rocket).

Name	noop	BIS	Cat	Ord	BCO
E-shop	94	53	94	99	36
Violet	20	21	22	28	23
Berkeley	15	9	14	19	12
T-shirt	20	11	19	14	11
rocket	327	157	139	144	143

Name	O	BO	BrO
Linux kernel	465	469	555

Table 8.3: Average explanation response times in ms

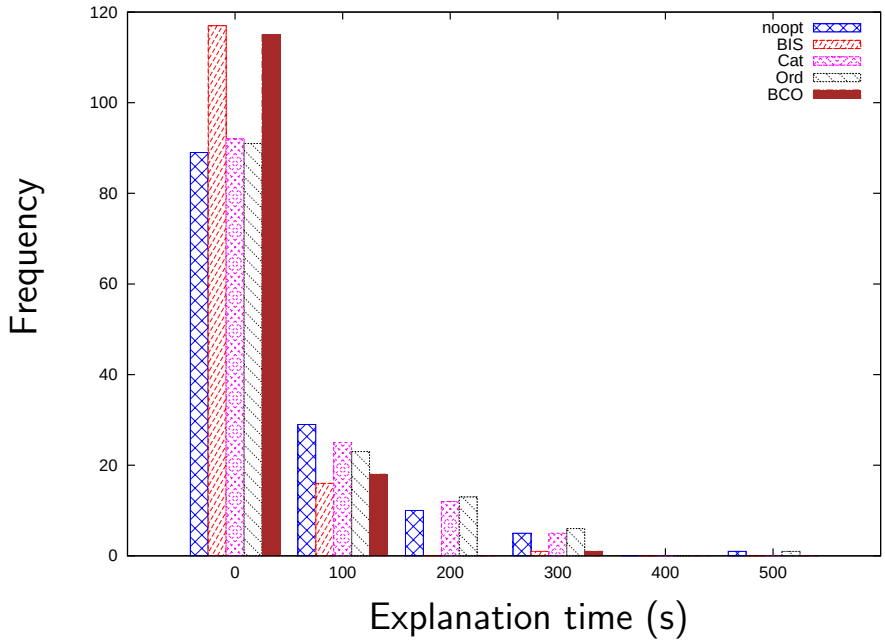


Figure 8.20: Distribution of explanation times for E-shop

8.6 Explanation Times

This section provides an overview of the observed response times for computing explanations. Table 8.3 presents the average response time for the individual instances and optimization combinations. Figures 8.20—8.25 depict the distributions of the times needed to compute explanations.

8.6.1 Discussion

The explanation response times are overall satisfactory. Even for the Linux kernel instance the average response times are predominantly around 0.5 ms. Somewhat surprising is the fact that the response times for the rocket instance

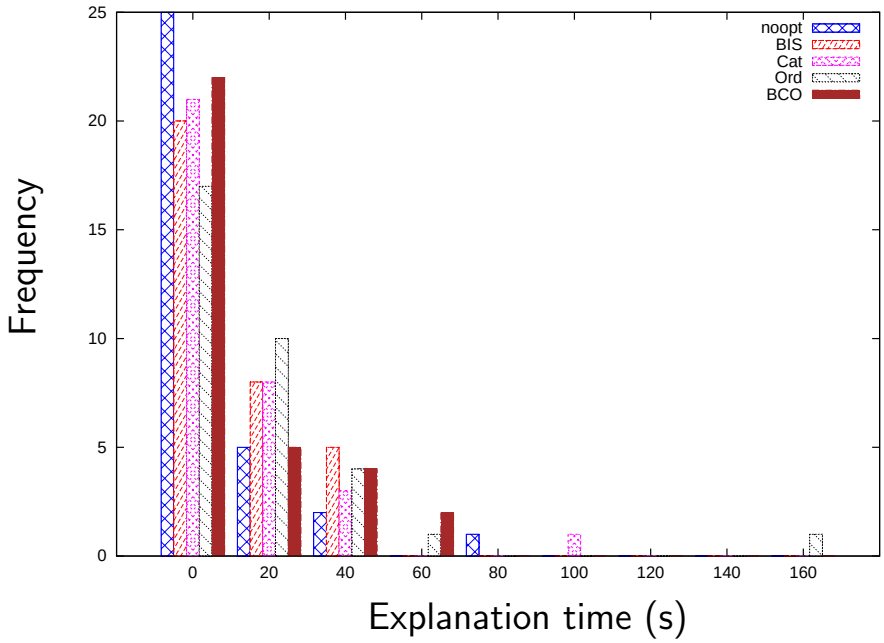


Figure 8.21: Distribution of explanation times for Violet

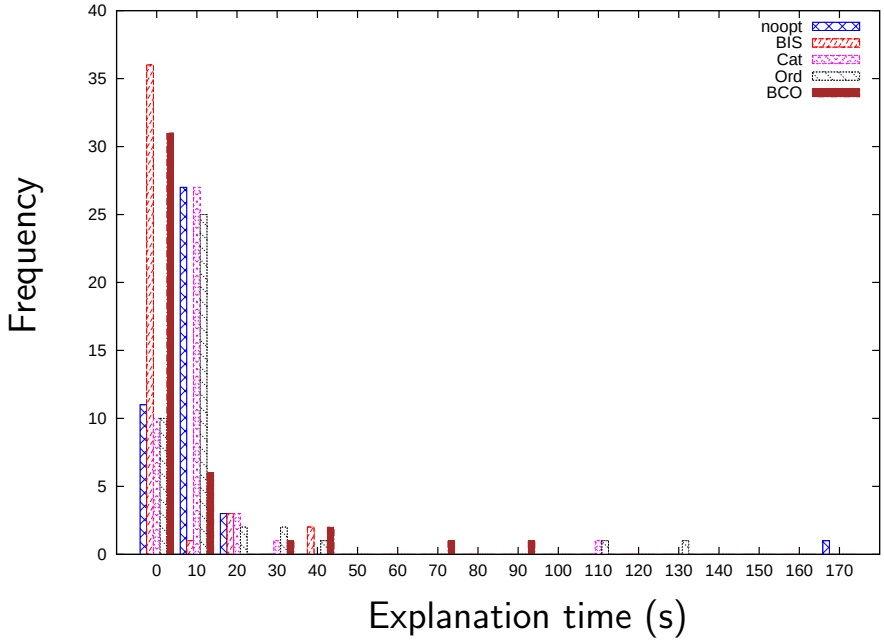


Figure 8.22: Distribution of explanation times for Berkeley

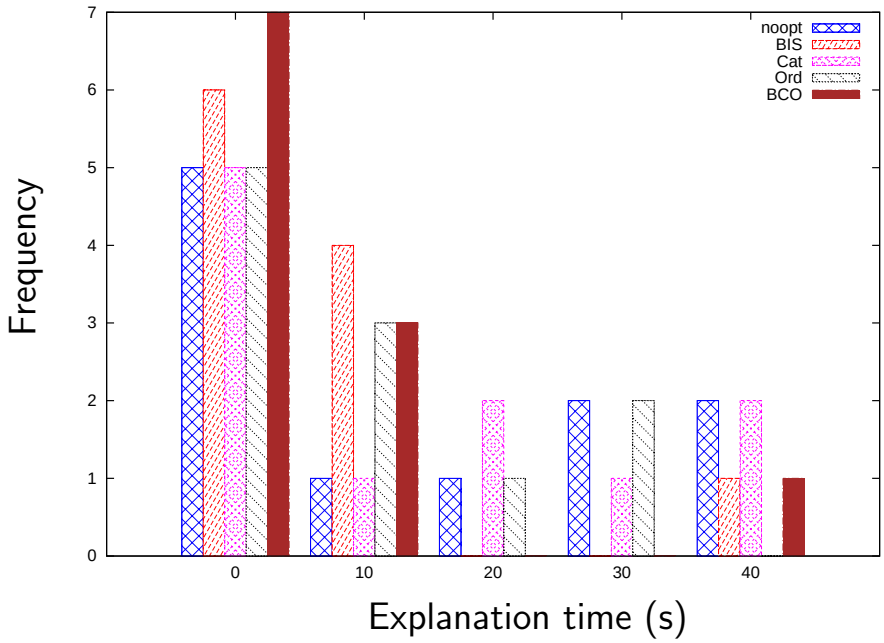


Figure 8.23: Distribution of explanation times for T-shirt

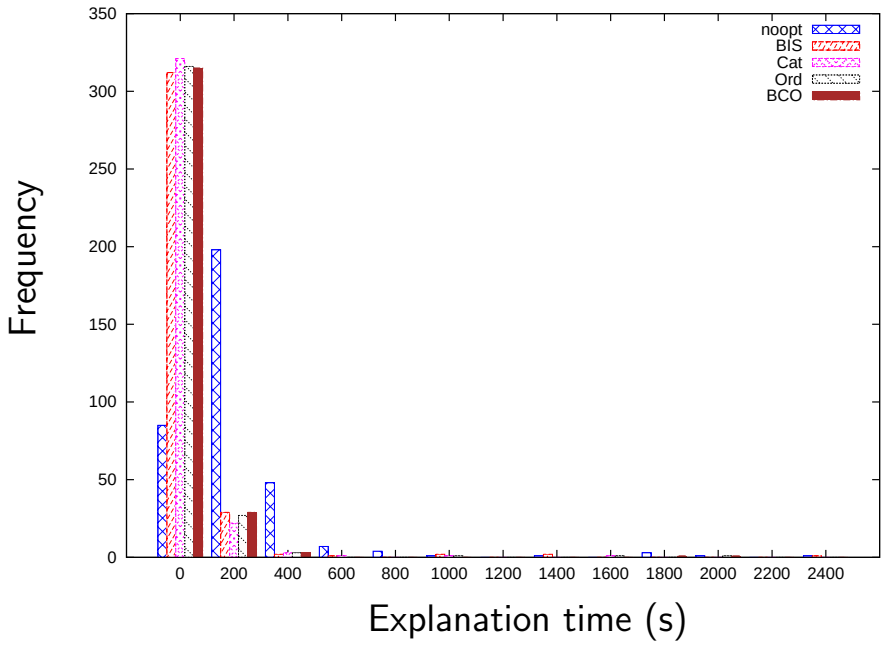


Figure 8.24: Distribution of explanation times for rocket

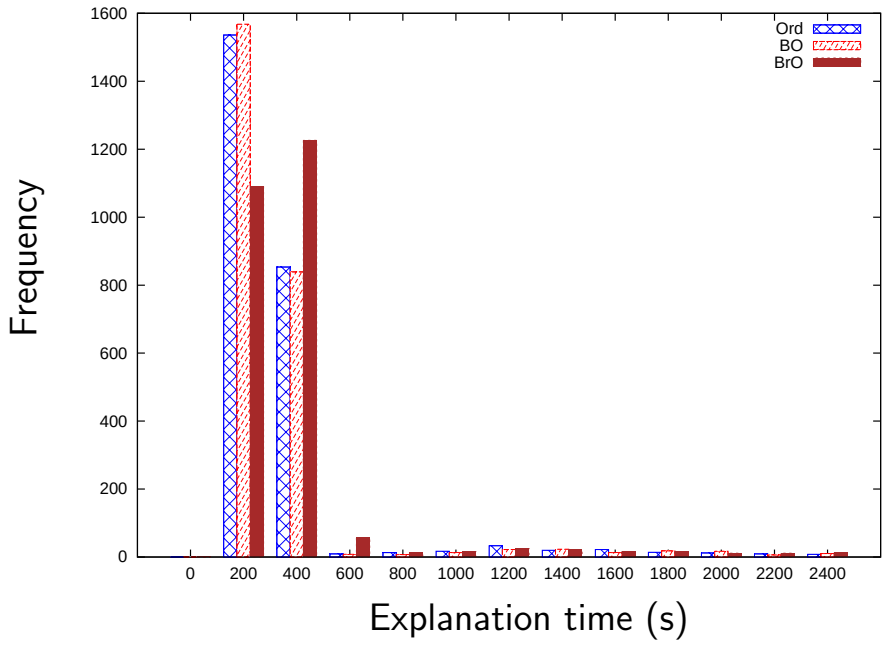


Figure 8.25: Distribution of explanation times for Linux kernel

are *shorter* than for the Linux kernel instance while the explanation sizes are *larger* (see Section 8.5).

As in the previous cases all the distributions have a long tail. Hence, in some (rare) cases the user has to wait much longer than what is the average response time. However, the worst-case times for explanations are much lower than for the inference of bound variables.

Interestingly, the BIS-optimization has a greater effect than in the inference response times in some instances. For instance, the time has been reduced by 50% for the E-shop instance. This indicates that the explanation algorithms are more sensitive to the size of the formula than the inference of bound variables.

In summary, the response times of explanations do not inconvenience the user, however, the appropriate algorithm for computing unsatisfiable cores needs to be selected.

8.7 Dispensable Variables Computation

There are two aspects of dispensable variables that are investigated in the presented measurements. One aspect is the response time of the proposed algorithm (Section 6.3). The second aspect is how much the technique of identifying dispensable variables helps to complete the process.

As in the case of measurements of the response times for inference of bound variables, the Linux kernel instance was measured separately. First, let us look at the small instances (E-shop, Violet, Berkeley, T-shirt, rocket, and 3blocks).

The time aspect is measured as for the response time all the inference of bound variables with the difference that only one version of optimizations is investigated; in particular the version **BCO**. The helpfulness of the technique is measured by recording the percentages of dispensable variables out of the variables that are not bound. In particular, if 100% of the unbound variables are dispensable, then invoking the shopping principle function binds all the unbound variables and consequently completes the configuration process.

Unlike in the case of computation of bound literals, the computation of dispensable variables sometimes does not succeed, it times out. For the purpose of the measurement, the timeout of 1 second was used. The motivation for choosing this particular timeout was twofold. Firstly, long waiting time for a technique that is supposed to speed up the configuration process defeats the purpose of the technique. Secondly, according to previous experience if the computation does not terminate quickly, then it takes unbearably long.

Interestingly enough, the only instance where some of the cases timed out was the E-shop feature model. A time out occurred in 1577 cases, which forms approximately 22% out of the 7215 cases investigated (Table 8.2).

The results of the measurements are again captured as distributions. For each instance there is an distribution of the percentages of dispensable variables. And, for each instance there is a distribution of computation times. However, here we present only the time distributions for the SAT instances, E-shop, and Berkeley since for all the other instances all cases were computed within 5 ms.

Computing Dispensable Variables for Linux Kernel Since the Linux kernel instance is significantly larger than the other evaluated instances, the timeout was increased to 3 seconds. However, despite of this increase the technique times out in most of the observed cases. Figure 8.28 depicts the distribution of response times and percentages of dispensable variables for the Linux kernel instance.

8.7.1 Discussion

The percentages of dispensable variables are somewhat surprising (Figure 8.26). All of the feature model instances have many dispensable variables in many cases. In the case of Berkeley the dispensable variables form 100% of the unbound variables most of the time and in the case of Violet it is even all the time. All features in this model are optional so such behavior is expected (see

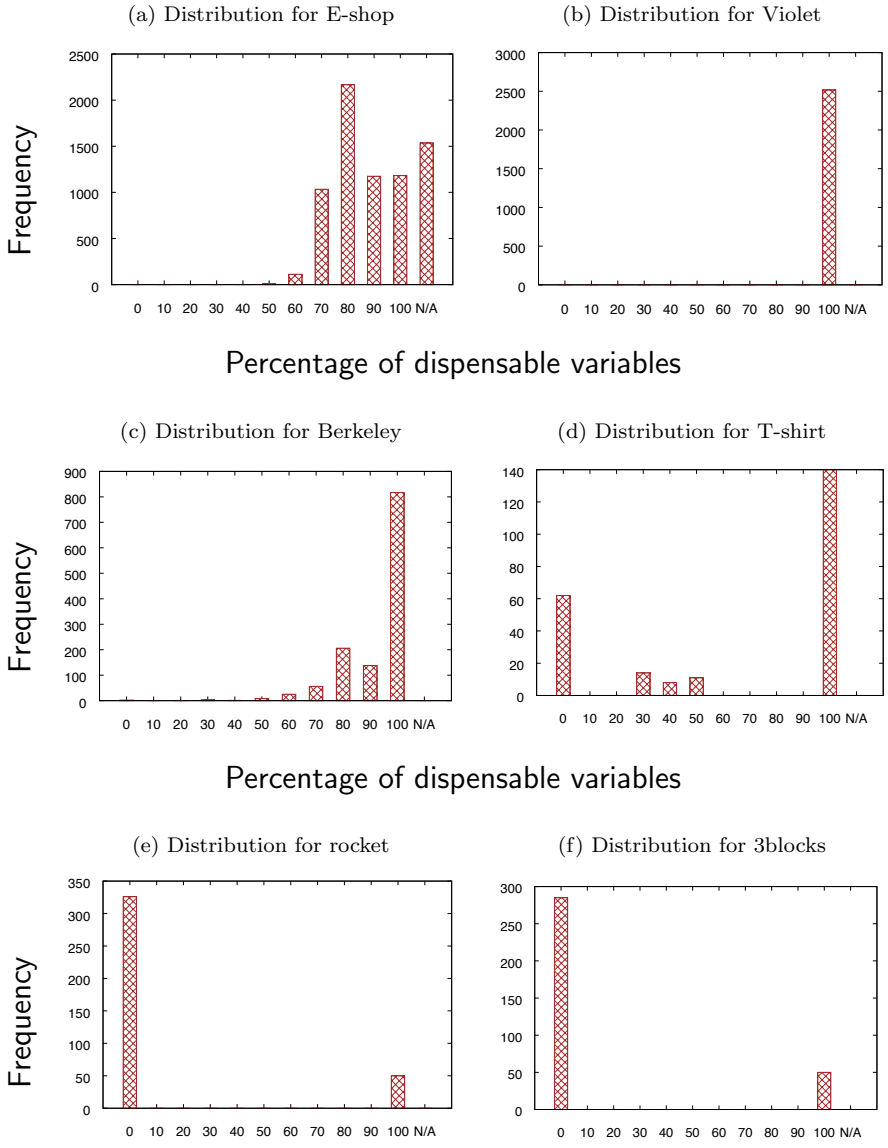


Figure 8.26: Distributions of percentages of dispensable variables

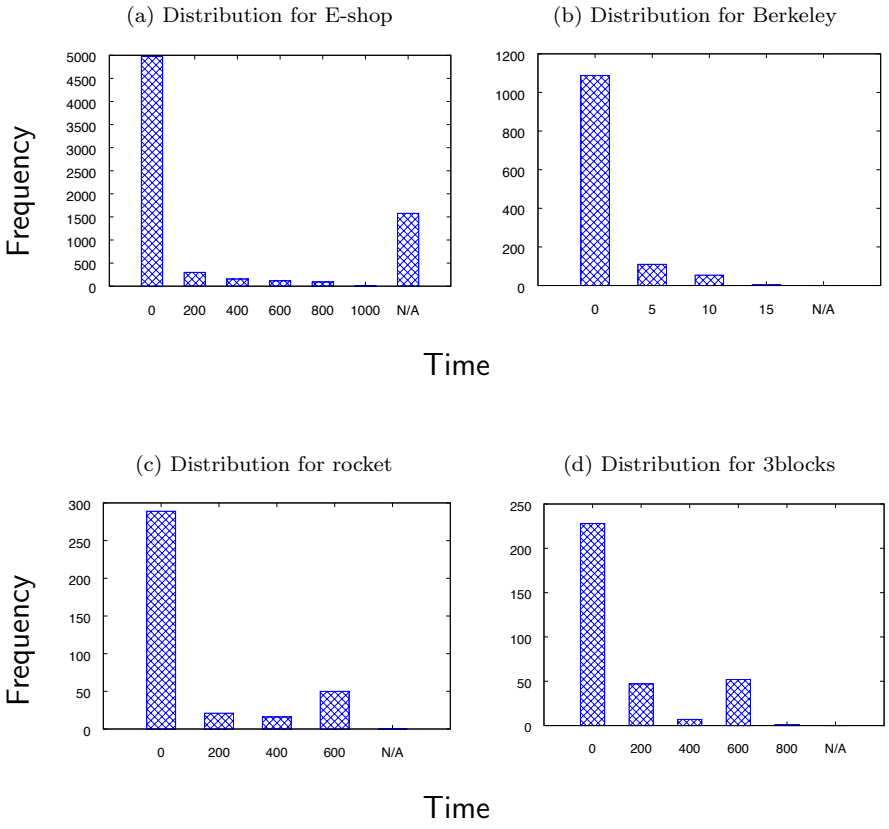


Figure 8.27: Distributions of response times for dispensable variables

also Example 32 for discussion about intuition of dispensable variables).

Altogether, invoking the shopping principle function often completes the configuration process for these two instances. The E-shop instance is harder to analyze because of the cases where the computation timed out. However, in the cases where the computation succeeded the percentages of dispensable variables where high as well: around 80% of the unbound variables were dispensable. Since the technique did *not* time out in 78% of the cases, this is a significant number.

Despite the fact that the instances from the SAT competition are larger in the number of variables and clauses than the feature model instances, the computation of dispensable variables succeeded in all cases. This illustrates the non-monotonic properties of minimal models (see Section 6.2.4). Recall that the computation of dispensable variables invokes the SAT solver as many times as there are minimal models of the state formula (Claim 10). Hence, these results stress that the number of minimal models cannot be easily estimated by looking

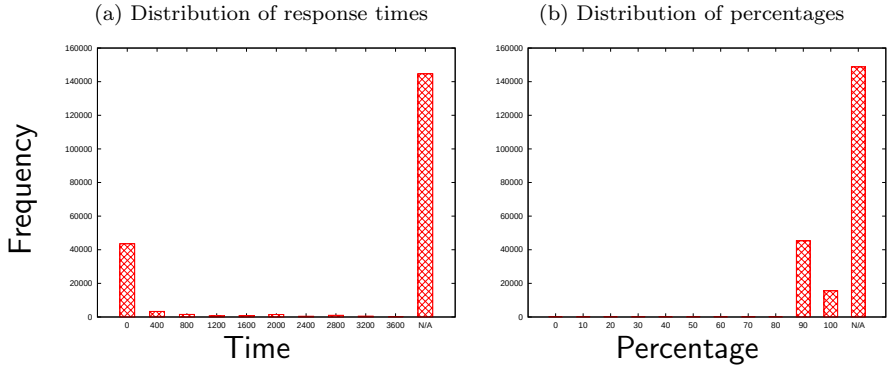


Figure 8.28: Distributions of response times and percentages of dispensable variables for Linux kernel with most cases timing out (the last column)

at the size of the formula.

Unlike the feature model instances, the SAT instances have very few dispensable variables. In fact, there were no dispensable variables in most of the cases. However, we should take into account that the configuration processes for these instances are very short and consequently have few unbound variables.

The Linux kernel instance shows that the technique scales poorly. In most cases the technique times out despite the fact that the timeout was increased to 3 seconds. Interestingly, large number of dispensable variables appears in the cases where the technique succeeds. However, since the number of cases where the technique succeeds is relatively low, this observation is little weight.

In summary the evaluation shows that the technique for computing dispensable variables suffers from poor scalability. This is due to the fact that it enumerates all minimal models. However, in the cases where the technique succeeds, the response times are short—less than 0.5 s in most cases.

8.8 Confidence

As the behavior of the user was simulated by a pseudo-random generator, we investigate how much confidence our measurements give us. In particular, whether the choices of numbers of measurements were adequate.

From a statistical perspective, a configuration process is a *walk* through a maze, where each door corresponds to a user decision. Each room has twice as many doors as there are unbound variables in that step (each variable can be either selected or eliminated). Hence, for an instance with n variables, each room of the maze has at most $2n$ doors. Each walk ends in a room with no outgoing doors, as all variables are bound at the end of the configuration process.

Name	noop	BIS	Cat
E-shop	53.76 ± 1.08	42.59 ± 0.78	53.96 ± 1.07
Violet	17.73 ± 3	16.74 ± 2.44	17.64 ± 2.73
Berkeley	N/A	16.46 ± 4.87	N/A
T-shirt	9.01 ± 0.65	7.97 ± 0.62	9.7 ± 0.82
3blocks	706.86 ± 63.39	663.88 ± 59.25	599.1 ± 50.97
rocket	185.42 ± 21.33	182.9 ± 21.17	176.05 ± 19.84

Name	Ord	BCO
E-shop	9.45 ± 0.07	7.98 ± 0.07
Violet	N/A	N/A
Berkeley	11.53 ± 5	8.41 ± 3.17
T-shirt	6.49 ± 0.55	N/A
3blocks	3649.35 ± 345.5	2847.86 ± 268.77
rocket	298.62 ± 39.51	285.08 ± 35.37

Name	O	BO	BrO
Linux kernel	1266.87 ± 56.66	1311.18 ± 62.23	1308.38 ± 68.86

Table 8.4: 99% confidence intervals for average response time in a walk in milliseconds. "N/A" marks cases that do not pass the Anderson-Darling normality test.

In our measurements we simulated 50 *random walks*.

To determine whether the chosen number is adequate, we investigate how the average response time differs from one walk to another. Intuitively, if the observed average times of evaluated walks were dramatically different, our confidence in the measurements would be low since a new walk would be more likely to be different from those that we evaluated. On the other hand, if the average times of the evaluated walks are close to each other, a new walk is likely to have similar characteristics to the ones that were evaluated.

To back this intuition with theory, we assume that the average response time of a walk is normally distributed over all possible walks and compute confidence intervals. In order to verify that that the averages are normally distributed *Anderson-Darling test of normality* [2] at 5% level[2] was used.

Since the test is recommended to be performed on a sample of less than 25 elements, 20 averages out of the 50 observed were sampled at random. A majority of the samples passed the normality test with a few exceptions (Table 8.4). However, the distributions breaking normality are all of short response times (less than 100 ms), where more erratic behavior is to be expected and precision is less of importance (for the time distributions and averages see Section 8.3).

[2] As is common in statistics, Anderson-Darling test tries to reject the hypothesis that the distribution is normal. The test produces a numerical value which indicates a deviation from the normal distribution. If this value exceeds a certain constant c_p, normality is rejected with the probability $1 - p$.

Name	maximal difference [ms]	maximal difference [%]
E-shop	1.46	4
Violet	7.56	10
Berkeley	14.68	14
T-shirt	3.56	39
3blocks	78.5	6
rocket	16.26	9

Table 8.5: The maximal difference between two averages of repeated measurements. One average is over 10 measurements and the second is over 7 measurements.

The confidence intervals are not provided for these exceptions.

Table 8.4 shows the 99% *confidence intervals* for the different versions and instances. The meaning behind these intervals is that, with 99% certainty, the expected average of a walk is within the interval. For instance, the first cell of the first table tells us that the expected average of a walk is within the interval $(53.76 - 1.08\,\text{ms}, 53.76 + 1.08\,\text{ms}) = (52.68\,\text{ms}, 54.84\,\text{ms})$. As we can see, virtually all the intervals are very tight. Only in the SAT-competition instances do the intervals go up to hundreds of milliseconds. In the feature modeling instances the intervals are mostly within the range of milliseconds. In the light of 99% guarantee, confidence intervals spanning only a number of milliseconds are clearly satisfactory. Moreover, obtaining such high confidence from only 50 walks, tells us that the walks are not dramatically different from one another.

The second aspect we investigate is the the number of iterations: except for the Linux kernel instance, each measurement was repeated 10 times. Since a large number of decisions were simulated, a sample of 250 was analyzed for each instance. Surprisingly enough, a large number of these 10-tuples did not pass the Anderson-Darling test for normality. Hence, applying confidence interval analysis on these observations is inappropriate.

A manual inspection of the observations led to the hypothesis that the observation in each 10-tuple oscillate in small variations. If this hypothesis is valid, there will be only small differences between averages of different samples. To support this hypothesis, for each instance and the 250 10-tuples an average of 7 elements and average of 10 elements was compared. If the difference is small, we can conclude that adding more measurements to the 10 performed ones will not significantly change the average.

Table 8.5 presents for each instance the maximal difference between the two averages that has occurred in the pertaining 250 10-tuples. The difference is presented as the absolute number of milliseconds and as the percentage with

respect to the average over 10 measurements.

We can see that in most of the instances the difference is less than 10% and less than 20 ms. The only instance with an excessive difference is the toy example T-shirt. That is not surprising because the response times for this instance are in units of milliseconds.

Hence, we can conclude that repeating each measurement 10 times is sufficient because adding more measurements does not significantly influence the value of the average.

8.9 Presenting Explanations

By automatically selecting or eliminating features the configurator provides comfort to the user and makes the configuration process less error-prone. However, there are cases where a user does not find a particular automatic choice satisfactory [159, 163, 199]. In such situations, the user may want to change some of the previous decisions in order to reverse the automatic choice. If the user is also the designer of the feature model, he or she may decide to modify the model. In either case, the user must understand *why* the configurator made the automatic choice in question.

This section compares different types of explanations and argues that explanations based on resolution trees (Section 4.5) provide greater level of information to the user.

Figure 8.29 presents a simplified feature model of a car where the user may pick between manual and automatic gearshift and between electric and gas engine. The feature model tells us that an engine must be *either* electric or gas and analogously the gearshift must be either manual or automatic. An additional cross-tree constraint tells us that an electric engine excludes manual gearshift (mutual exclusion).

The figure depicts a situation where the user selected the manual gearshift. The selection of this feature together with the semantics of the feature model determines the values for all the other features. Let us focus on the following question asked by the user:

Why is Gas *automatically selected?*

Humans typically reason from the consequence in question towards the root causes. In the case of feature models, this means starting from the feature in question and by applying the semantics of feature models going toward user decisions. Using this reasoning, the explanation for automatic selection of Gas takes the following form.

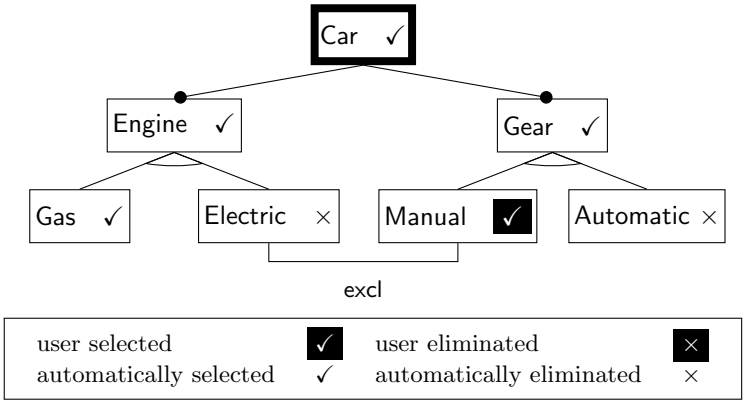


Figure 8.29: Simple feature model of a car with Manual selected by the user

- Gas is selected because one of Gas or Electric must be selected and Electric is eliminated.
- One of Gas or Electric must be selected because they are in an alternative subgroup of Engine.
- Electric is eliminated because it is excluded by the selection of Manual.
- Engine is selected because it is a mandatory sub-feature of the root feature Car.

Note that in English we typically say "*chain* of reasoning" or "*chain* of thought". However, rather than in a chain, this process is organized in a *tree*. For instance, the user needs to understand why Engine is selected and why Electric is eliminated. Since the reasons for these two effects are independent, the "chain" splits into two independent branches.

This rather small example already shows that a user needs to consider a number of steps in order to understand why the feature is automatically selected. This represents significant effort for the user and hence warrants mechanized support.

Let us investigate the level of information provided by the different explanation techniques (Section 4.5).

Precompilation techniques use as an explanation the user decisions that has led to the automatic selection of the feature in question. For our example this highlights a single feature as that is the only user decision made for the feature model (Figure 8.30). In this case, such explanation provides insignificant information. In more general cases where multiple user decisions are present, this form of explanations may be useful. However, the user still needs to come

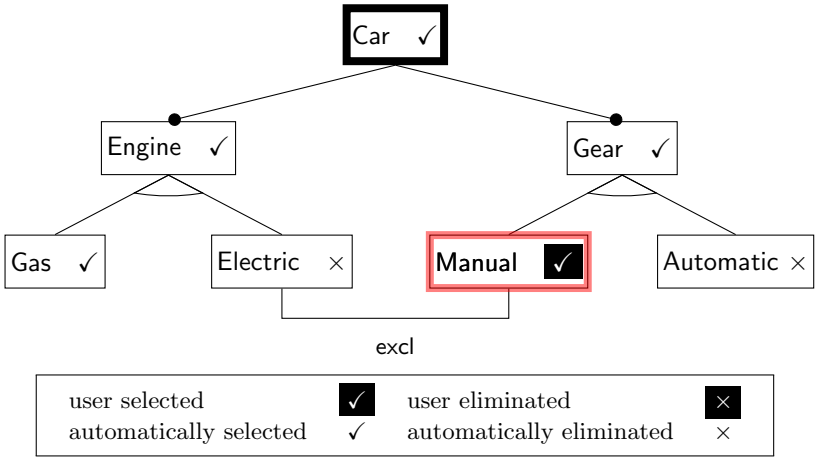


Figure 8.30: Simple feature model of a car with Manual selected by the user where elements responsible for the automatic selection of Gas highlighted. The explanation is in the form of pertaining user decisions. In this case it is the single Manual that is highlighted as an explanation.

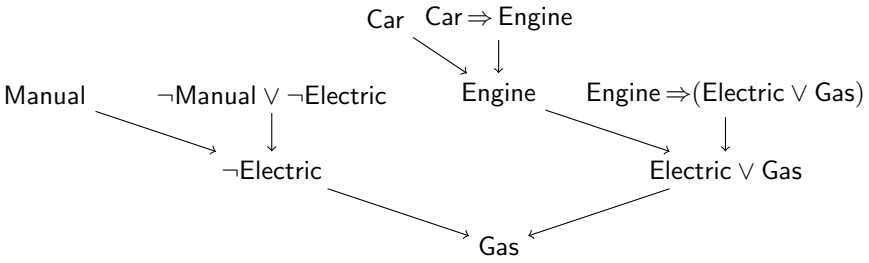


Figure 8.31: Resolution-based explanation for Gas being selected automatically in the form of the semantics of the feature model

up with the pertaining relations between the modeling primitives of the feature model in order to fully understand why the automatic choice has been made.

8.9.1 Resolution Trees

Section 4.5 proposes the use of resolution trees as forms of explanations. First, let us look at the resolution tree for our question: Why is Gas selected (Figure 8.31)? The resolution tree is close to the "tree" of thought outlined above. The clause Gas is obtained by resolving ¬Electric and Electric ∨ Gas. The clause ¬Electric corresponds to elimination of Electric and Electric ∨ Gas corresponds to the fact that at least one of Electric, Gas must be selected. Following the resolution steps towards the leaves in an analogous fashion, all nodes are gradually explained.

The resolution tree itself is already a helpful information for the user but

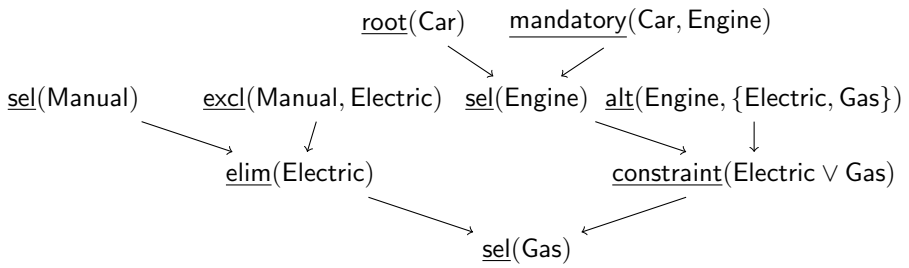


Figure 8.32: Resolution-based explanation for Gas being selected automatically in the form of modeling primitives

the user interface has a number of options how to present it in that more user-friendly fashion.

To bring the user closer to the original feature model, the resolution tree, which is in the form of the semantics of the model (Figure 8.31), can be translated into a tree of corresponding feature modeling primitives (Figure 8.32). Such tree is obtained by replacing each clause with the modeling primitive that generated the clause (see Chapter 7 for the realization of the mapping). The tree conveys the same information as the original resolution tree but does not require the user to understand logic notation.

8.9.2 Presenting Resolution Trees

The disadvantage of resolution trees presented so far (Figure 8.31 and Figure 8.32) is that they are detached from the original feature model. Highlighting in the feature model those primitives that appear in the resolution tree addresses this disadvantage (Figure 8.33). In order to understand why Gas is automatically selected, a user follows the primitives marked red and recalls their respective semantics. The disadvantage of this representation is that the order of consequences is lost.

Projecting the resolution tree on the feature model addresses this disadvantage (Figure 8.34). This representation combines all the information in one diagram. Note that an additional node Gas ∨ Electric was added to represent an intermediate stage of the reasoning[3]. Such explanation is straightforward to interpret. Each non-leaf node that is part of the explanation is a consequence of two reasons. These reasons are those nodes that point to the consequence via red arrows. For instance, one can immediately see that Electric is selected because Manual is selected and there is an exclusion edge between Electric and Manual.

[3]Such nodes can be removed for less advanced users.

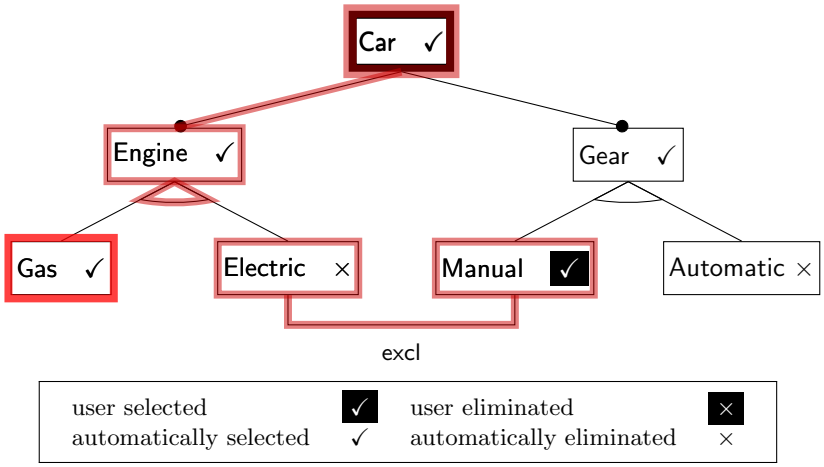


Figure 8.33: Simple feature model of a car with Manual selected by the user where elements responsible for the automatic selection of Gashighlighted

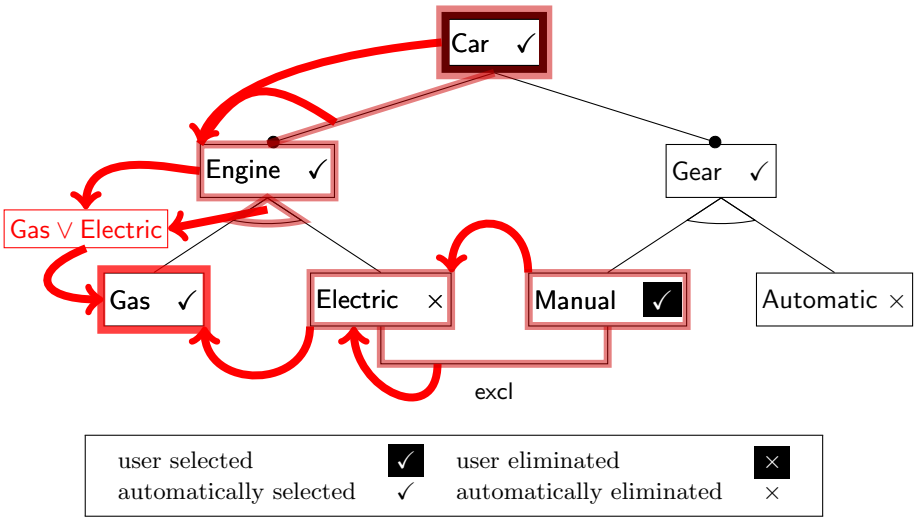


Figure 8.34: Resolution-based explanation for Gas being selected automatically projected on the depiction of the feature model. The additional node Gas ∨ Electric represents intermediate information not contained in the original feature model.

8.9.3 Summary of Explanation Presentation

This section presents the advantages of using resolution trees to provide explanations to the user over just highlighting pertaining user decisions, a technique used in precompilation methods.

Highlighting the pertaining user decisions that are responsible may represent a small part of the information that is needed to understand why a certain feature is automatically selected or eliminated (Figure 8.30).

Presenting explanations as resolution trees enables reasoning akin to the reasoning that humans use when looking for a root-cause. A user starts from the feature to be explained and gradually proceeds toward the pertaining user-decisions.

The user interface has a number of options how to present resolution trees to the user. The actual resolution tree at the semantic level provides the most detailed information but requires the user to be familiar with logic (Figure 8.31). Transforming such resolution tree into a tree of the pertaining modeling primitives conveys the same information but requires only the knowledge of the modeling primitives (Figure 8.32).

The explanation can be shown on the feature model diagram by highlighting modeling primitives that appear in the resolution tree (Figure 8.33). Projecting the resolution tree on the feature model diagram additionally shows reasons and consequences in the explanation (Figure 8.34). We argue that this last representation is the most informative one because it shows all the information in a single diagram.

8.10 Conclusions and Summary

This section discusses the observed behavior from the perspective of the following standards coming from the research on acceptable response times [98]:

- 0.1 seconds is classified as instantaneous,
- 1.0 seconds as uninterrupted user action that is noticed by the user but does not hinder the work, and
- 10 seconds is a limit for keeping user's attention.

Bound Variables Response Times

For instances with hundreds of variables, we have seen that when computing bound variables, the configurator responds in 0.1 seconds in many cases and

takes over 1 second only in a few cases for hard instances. However, long times—over 10 seconds—may appear if a wrong heuristic is applied. In particular, the **Ord** heuristic does not perform well for the instances from the SAT competition, which are far more constrained than the variability models.

The evaluation of computation of bound variables for the Linux kernel shows that the presented approach is applicable in instances with thousands of variables. The response times for this instance are predominantly around 1 second. Longer response times appear—up to 10 seconds in several cases. However, such times are extremely rare—response time exceeded 5 seconds in less than 0.02% of the inspected cases.

Explanations

The evaluation of explanations for variability models is also satisfactory. The response times are less than 0.5 second for most of the cases with some rare exceptions of 2 seconds, which occurred in the Linux kernel instance. The explanation sizes are relatively small. In the largest variability model instance (Linux kernel) the average explanation size is 16, with some rare exceptions of the size 100 clauses. However, the evaluation shows that the QuickXplain algorithm implemented in the solver SAT4J does not scale well and the above results for the Linux kernel instance were obtained with the tool MUSER (see Section 7.1).

The evaluation of explanations for the instances from SAT competition show that the sizes are unwieldy for such types of instances, i.e., instances that are tightly constrained (the number of clauses is much higher compared to the number of variables). The sizes of explanations for these instances exceed 2^{32}, which is clearly incomprehensible for a human. In general, such large explanations are most likely inevitable due to the result by Cook and Reckhow according to which proofs are super-polynomial in size of the original formula unless *co-NP*$\neq$*NP* [42]. Nevertheless, the variability models do not exhibit such excessive explanation sizes.

Section 8.9 demonstrates the usefulness of resolution trees in explanations already on small examples. The evaluation of the explanation sizes shows that in real-world instances we can expect explanations counting up to 30 elements and average around 15 elements. Together these observations show that resolution-based explanations help the user in many cases.

BIS-optimization

The BIS-optimization shows promising improvements of response times in several instances. Unfortunately, the response times for these instances are already

short without the optimization. In the large instances (Linux kernel) where improvement of the response times would be useful, the optimization does not help.

However, the optimization is interesting from the perspective of explanation sizes. In some cases reconstructing the explanation from an explanation on the optimized formula yields a smaller explanation than when the optimization is not used. This indicates that in some cases the explanation algorithms work significantly better on smaller formulas.

Dispensable Variables

The analysis of the dispensable variables technique proved the technique to be fast for instances with hundreds of variables—the response time was below 200 ms in most cases. However, there are some cases where the proposed computation approach fails to deliver results. The technique helps the user to bind many variables in many cases. In fact, there were many cases where all of the unbound variables were dispensable, i.e., cases where the shopping principle function completes the configuration process.

The Linux kernel instance shows that the technique for computing dispensable variables does not scale well as it times out in most of the cases. Hence, a different technique needs to be developed for variability models with thousands of variables.

Confidence

The analysis of the response times using the methods of confidence intervals shows that the evaluated sequences differ little from one another. This is important from practical perspective because it means that it is possible to accurately estimate the response times from a small number of observations.

All of the observed distributions have the common property of long tails, i.e., extreme values occur very rarely. This is a price for the lazy approach (computation is performed only when needed). However, these tails encompass small number of cases.

Chapter 9

Thesis Evaluation

The statement to be investigated by this dissertation is the following:

> *SAT solvers are better for implementing a Boolean interactive configurator than precompiled approaches.*

In this chapter we argue that the thesis statement is validated by the presented material. Since we are in the domain of propositional constraints, the precompiled approaches are represented by BDDs and we show that the advantages of SAT solvers over BDDs are the following:

- SAT solvers scale better,
- SAT solvers are better for providing explanations,
- the response times of the configurator do not make a significant difference, and
- there is more potential for further research in SAT-based configuration.

These arguments are detailed, in turn, in the following sections.

9.1 Scalability

Examples in Literature

BDDs enabled significant progress in model checking in the early 90s [35, 144]. However, the issue of scalability of BDDs motivated research in the application of SAT solvers in model checking. For instance, Abdulla et al. recognize that BDDs are typically unusable for systems with more than hundreds of variables [1]. Biere et al. provide an empirical case study where the SAT-based checker consistently outperforms the BDD-based one [23].

Similar trends are appearing in analyses of feature models. Mendonça et al. study heuristics for compiling feature models into BDDs. The need for such heuristics is motivated by the poor scalability of BDDs [148]. However, in a later publication, Mendonça et al. show that SAT solving in feature models is easy[1] [147]. Hence, the motivation for studying heuristics for construction of BDDs is diminished, since a SAT solver can be used instead. It should be noted, however, that for some types of analyses, BDDs give better results; an example of one such analysis is the counting of all possible solutions of an instance [20].

Theoretical Evidence

As mentioned in the background section, the size of a BDD depends on the ordering of variables in the tree's branches (Section 2.2.2). It is well-known that deciding whether a variable ordering is optimal or not is NP-complete [26]. Also, exponential lower bounds are known for the size of BDDs [33]. This further discourages the use of BDDs, since large amount of data needs to be included with the instance to be configured.

Despite the empirical evidence that SAT solvers are applicable in more cases in practice than BDDs, theoretical understanding of such is limited. Nevertheless, there are some theoretical results that indicate that constructing a BDD is, in general, significantly more difficult than deciding satisfiability. The argument stems from the fact that having a BDD enables us to calculate *in linear time* the number of models of the pertaining formula, and, calculating the number of models is a #P-complete problem[2] [194, 162]. The difficulty of solving a #P-complete problem is illustrated by Toda's theorem, whose direct consequence is that a Turing machine with a #P oracle can solve any problem in polynomial hierarchy in polynomial time [191]. Hence, a BDD oracle enables us to answer any problem in polynomial hierarchy in polynomial time. In contrast to that, problems already in the second level of polynomial hierarchy require super-polynomial (in practice exponential) queries to a SAT oracle. While this is not a formal proof of the fact that constructing a BDD is harder than SAT, it lends credence to such hypothesis, as there is a significant difference in the power of the corresponding oracles.

[1] The notion of easy and hard SAT instances is based on a well-known phenomena that the hardness of an instance depends on the ratio of the number of variables to the number of clauses [155].

[2] #P is a class of problems that are defined as computing the number of accepting paths of a nondeterministic Turing machine that is running in polynomial time.

Evaluated Instances

The conducted experiments show that the SAT-based approach successfully copes with all the evaluated instances. In contrast to that, construction of a BDD from the Linux kernel instance did not succeed even after 24 hours of computation time using a tool provided by Grigore [170]. As mentioned in the introduction, the fact that the particular tool did not succeed building the BDD does not mean that it is impossible. It might be that some heuristics specific to that instance will succeed in precompiling it. However, the SAT-based approach *scales uniformly*, as no heuristics specific to the instances were applied. Thus, the SAT-based approach is more practical as it imposes fewer requirements on the user.

Another argument for scalability of the SAT-based approach is the growth of configurator's response times that is observed in the evaluated instances. In feature models with hundreds of variables the times were in the range of 0–200 ms, while they are predominantly in the range of 0.5–2 s for the large example with over eight thousand variables. This means that the times *grew linearly* with the size of the problem, rather than exponentially as the complexity class of the problem suggests. This further corroborates the above-mentioned findings of Mendonça et al. that the SAT problem is easy for variability models [147].

9.2 Providing Explanations

We have shown how explanations are produced from resolution-based proofs (Section 4.5). Compared to existing work, the provided explanations are more informative since they comprise parts of the configured instance and dependencies between them. The following paragraphs summarize the capabilities of SAT-based solutions and the issues with BDDs from the perspective of resolution proofs.

To produce resolution-based proofs for state-of-the-art SAT solvers is natural [206]. In fact, a number of popular solvers produce such proofs [167, 151, 107]. Some solvers do not produce a resolution tree, but instead produce an *unsatisfiable core* (unsatisfiable subsets of the given clauses) [111, 140, 142]. In order to enable constructing a tree from such unsatisfiable core, we provide a SAT solver that constructs a tree once invoked on the core [102].

Existing research shows that it possible to produce proofs of unsatisfiability from BDDs [186, 112]. However, there are several issues with this technique.

- The resulting proof is not a standard resolution tree, but instead it is an *extended resolution tree* (using the operations of *extended resolution*) [192]. It is not clear how to compute explanations from such proofs.

- The extended-resolution-based proof is obtained from a BDD certifying unsatisfiability. Such a tree comprises a single node labeled with FALSE. However, the algorithms for interactive configuration based on BDDs do not explicitly construct such BDD [92, 189, 88, 187, 188]. The complexity requirements for constructing such tree are unclear, especially in the presence of proof construction.
- The commonly-used BDD libraries do not support such proof generation.

Hence, when using BDDs in interactive configuration, the explanations can only be given in the form of user decisions responsible for the queried locked variable. This gives little information to the user as it is necessary to also consider the configured instance in order to understand the reason for the lock (see section on explanations for further details Section 4.5).

9.3 Response Times

The research on applying BDDs to interactive configuration is often motivated by *guaranteed response time*, i.e., once a BDD is built, the response time is known, since the algorithms providing feedback in interactive configuration require time linear to the size of the BDD. We argue that this motivation is of little importance as the times obtained in our measurements for the SAT-based approach were predominantly within the range of hundreds of milliseconds for the models for which BDDs compile. According to research on human interfaces, such response times do not pose any inconvenience upon the user [98]. Moreover, even though the SAT-based approach does not provide a theoretical guarantee for the response time, the empirical evaluation shows that the response times for user decision sequences differ very little from one sequence to another (Section 8.8). Hence, it is possible to estimate the response time for a particular instance with high accuracy.

The Linux kernel variability model is currently the largest variability model available in the research community. Even in the case of this large instance, the response times were predominantly in the range of 0.5–2 s. According to research on human interfaces, such times are acceptable for the user—Hoover classifies such times as *"uninterrupted user actions, they will notice a delay but can continue to work"* [98]. Finally, we believe that these times can be improved by reengineering—for instance, by using a more efficient programming language than Java for the implementation.

9.4 Potential for Further Research

While it is difficult to provide hard evidence that there is more potential for further research in the SAT-based approach than in the BDD-based approach, we argue that it is indeed the case.

SAT solvers are used as basis for numerous solvers for non-propositional logic. Such solvers either by translate the given problem into propositional logic or extend a SAT solver (see Section 10.5 for further details). We conjecture that the techniques used in such solvers can be used to extend our approach to interactive configuration of instances with non-propositional semantics. A prominent example are pseudo-Boolean constraints, which are constraints that enable us to express dependencies such as maximal price or weight (see Chapter 11 for further discussion).

There are numerous opportunities for response time improvements via syntactic optimizations that are commonly used in SAT solving (see Section 10.2 for further references). We have shown how to integrate the BIS-optimization into a configurator. We have shown that the optimization in some cases leads to improvements in response times of up to 20%. We conjecture that it is possible to achieve further improvements by integrating other syntactic optimizations.

9.5 Summary

We argue that the presented work validates the conjecture that using a SAT solver is better for implementing a Boolean interactive configurator than using a precompilation approach. In particular, the SAT-based approach scales better and enables more informative explanations. The presented measurements show that the motivation for the precompilation approach, i.e., guaranteed response time, is of little importance due to the following facts:

- The response times observed in the SAT solution are satisfactory.
- Different inputs from the user produce highly similar response times.

Hence, while the SAT-based approach does not provide a theoretical guarantee for response times, in practice it is possible to estimate the response times with high accuracy, and, the response times observed on the evaluated instances do not inconvenience the user.

Consequently, from the perspective of a programmer that is implementing a configurator, the SAT approach is more practical. To obtain good response times for instances counting hundreds of variables, it is sufficient to use a simple algorithm that requires no heuristics and uses the SAT solver as a black-box. To obtain good response times for instances counting thousands of variables, it is

sufficient to slightly modify the underlying SAT solver. In contrast to that, using precompilation with large instances requires specific heuristics or modifications of the underlying data structure. Finally, in both types of instances large and small, SAT solvers enable more informative explanations.

Chapter 10

Related Work

The related work is organized according to the order of the chapters in this dissertation. Section 10.1 overviews the research done in configuration with the emphasis on interactive configuration. Section 10.2 overviews research in formula optimizations akin to the BIS-optimization (Chapter 5). Section 10.3 discusses research related to the chapter on the completion of a configuration process (Chapter 6) and in particular to research on the computation of dispensable variables (Section 6.3). Section 10.4 points to research related to the chapter on the construction of the configurator (Chapter 7).

10.1 Configuration

There is a close relation between configuration and backtracking. In the case of backtracking, an algorithm is exploring the state-space in order to find a solution to the given problem. Backtracking algorithms often employ heuristics to determine where to look first [173] and propagation techniques to avoid parts of the state-space that certainly do not contain a solution [122]. In configuration, it is the user who determines the order of search and the configurator's task is to warn the user about decisions leading to parts with no solution.

From this perspective, configurators are similar to Assumption-Based Truth Maintenance Systems (ATMSs) introduced by de Kleer [52]. An ATMS is a generic data structure to be used with a backtracking algorithm. An ATMS is used by the algorithm to store information obtained during the search so it can be reused later to avoid duplicate calculation and to avoid sub-spaces without solutions.

Another correspondence between configuration and backtracking is the concept of backtrack-freeness, which originally comes from research on backtracking by Freuder [75]. A CSP problem is backtrack-free if there exists an ordering

of the variables such that any algorithm assigning values in this order while preserving consistency of what has been assigned so far, always finds a solution. The notion of backtrack-freeness in the context of configuration (see Section 3.1) is stronger because a backtrack-free configurator must ensure this property for *any* ordering that the user might choose.

The concept of configuration or interactive configuration appears in multiple domains. Probably the oldest flavor of configuration is *product configuration*. In product configuration the task is to assemble a product from a set of components which can be connected through their ports while some constraints must be respected. The bulk of research on product configuration deals with cataloging the components but such issues are not considered in this dissertation. See an extensive survey on product configuration by Sabin and Weigel for further reference [175].

Lottaz et al. focus on configuration of non-discrete domains in *civil engineering* [131]. They have applied Fourier-Motzkin elimination to support a civil engineer in the design of an apartment. Such design must conform to certain constraints, e.g., minimal sizes or distances. These constraints are captured as linear inequalities. Additionally, non-linear constraints are supported through approximation.

Another research on interactive configuration of non-discrete domains was done by Hadzic and Andersen [90]. Compared to Lottaz et al., they take into account optimization constraints and apply the simplex method [169]. The motivational example they use is the diet problem—a problem of finding a diet plan for soldiers with minimal cost and which satisfies certain dietary constraints.

Despite of the promising results of the two promising works referenced above, it seems that configuration of non-discrete domains is rarely needed. This especially true in configuration of software where domains are typically not only discrete but also finite (some exceptions to this are mentioned below).

Finiteness and discreteness of make the search space amenable to *precompilation*. The motivation for precompilation is that if a constraint is compiled into a specialized data structure, some computationally hard problems become easier [49]. In particular, Vempaty showed that a CSP can be compiled into a finite automaton, often referred to as the *Vempaty automaton* [197].

Building on Vempaty's research, Amilhastre et al. show how to perform interactive configuration on a CSP by first compiling it into a Vempaty automaton [3]. Other authors take similar approach. Fargier and Vilarem use *tree-driven automata* instead of general finite automata to reduce the size of the automaton [65]; Madsen uses *uniform acyclic networks* [136]. Pargamin studies precompilation into *cluster tree* scheme in the presence of objective function maximization or optimization in order to configure a CSP [164].

One of the most popular data structures for precompilation are Binary Decision Diagrams (BDDs) [34]. In general, the size of a BDD may be intractable. Moreover, finding the smallest BDD is NP-hard [26]. However, both theoretical [86] and experimental [193] research shows that on some problems BDDs perform better than resolution-based provers but also the other way around.

> ***Remark 10.37.*** *BDDs are a special case of Vempaty automata because Boolean formulas can be seen as CSPs where all the variables have the domain* {TRUE, FALSE}*. However, BDDs are more attractive for practitioners for their support in the form of libraries (for instance [109, 108]).*

Hadzic et al. show that once a propositional constraint is represented as a BDD, interactive configuration can be done in time polynomial with respect to the size of the BDD [92, 189, 88]. In order to cope with the sizes of BDDs, Subbarayan et al. use *trees of BDDs* [187] and *Join Matched CSPs* [188].

Hadzic and Andersen also study global constraints, such as total price, in the context of propositional configuration and BDDs. The problem they tackle is to perform configuration while respecting an upper limit on the total cost [89]. Andersen et al. extend the approach to deal with multiple cost functions at the same time [5].

Van der Meer et al. show that BDDs can be used to implement backtrack-free and complete configurators even for a certain type of unbounded spaces. The motivational example they use is a support for constructing component architectures with arbitrary number of components but that must conform to some overall constraints [196].

Another example of configuration of unbounded spaces is the configuration of strings. Hansen et al. show that regular string constraints can be configured using finite automata [93, 94].

Ensuring backtrack-freeness is in general computationally expensive. In response to that, some configurators are not backtrack-free, i.e., the user may run into a conflict with the original constraint (see Chapter 3). Batory uses unit propagation (see Section 2.1.3) to provide feedback to the user [15].

Freuder uses *arc consistency (AC)* (see Section 2.1.4) in the context of CSP [79]. It is worth noting that AC leads to backtrack-free configuration for certain types of constraints as shown by Freuder [75]. Freuder et al. also show that backtrack-freeness can be ensured in the context of AC by removing certain solutions, i.e., sacrificing completeness of the configurator [77].

Both of these techniques, unit propagation and AC, are not backtrack-free but are computationally cheap and enable providing nice explanations to the user. These explanations can be trees similarly as done in this dissertation (see

Section 4.5). The approach of Batory and the approach of Freuder are rather similar due to the relation between unit propagation and arc consistency (see Section 2.1.5).

If a configurator is not backtrack-free, then the user may run into a conflict—a situation where the user decisions violate the configured problem. If that is the case, it is desirable for the configurator to provide support for amending the conflict. This can be done by *constraint relaxation* (see for instance [163]) or by *corrective explanations* (see [159, 199]). Constraint relaxation tries to remove constraints to restore consistency, while corrective explanations try to change user decisions, as little as possible, to obtain a consistent set of decisions; corrective explanations are related to the MostClose problem defined by Herbrard et al. [95].

The research discussed so far is concerned with configuration that yields one configuration from a space known beforehand. However, there is also research on the *synthesis* of the outcome. For instance, O'Sullivan uses interactive constraints propagation in *conceptual design* [161].

Interactive configuration has also been applied in the context of transformation. Janota et al. show how interactive support can be used to transform a propositional logic semantics into a feature model (using BDDs) [106]. The difficulty in this transformation is that one semantics may correspond to multiple feature models [48]. The presented solution enables the user to gradually specify the particular feature model he or she wants.

It should be noted that the research on configuration listed above is inevitably incomplete. The problem of configuration seems to be fundamental as it may be found even in areas rather distant from software engineering, for example in language parsing [64].

10.1.1 Bound Literals

For a configurator to be complete and backtrack-free, it must identify bound literals, i.e., literals that evaluate to True under all satisfying assignments of a formula (see propositions 3 and 2). Bound literals are not only important in interactive configuration. In the research on the complexity of the SAT problem, bound literals are known under the name *backbones* [156, 27, 118]. The correspondence between satisfiability and backbones (see Proposition 1) has been exploited by Fox et al. [85].

Bound literals are also important in non-interactive configuration. Kaiser and Küchlin use the terms *inadmissible* and *necessary* variables. A variable v is inadmissible iff the literal $\neg v$ is bound. Conversely, a variable v is necessary iff the literal v is bound. Kaiser and Küchlin use an algorithm similar to Test-

VARS to detect bound literals [113].

10.1.2 Proof Size Reduction

As mentioned in Section 4.5, it is desirable to provide explanations as small as possible since it makes them easier to understand. The explanations are constructed from the proofs of unsatisfiability, and, these proofs are in the form of a resolution tree.

As a side note, we mention that the sizes of proofs have a theoretical relevance in particular to the relation *co-NP≠NP* shown by Cook and Reckhow [42]. However, since this dissertation is predominantly practical, we refer to practical proof reduction techniques here.

Ben-Sasson and Wigderson define the *width of a resolution tree* as the number of literals in the largest clause appearing in the tree and they show the relation between tree's width and its size. Further they propose an algorithm, based on exhaustive resolution, to derive a resolution tree with the smallest width [19]. This approach, however, is not directly applicable in modern SAT solvers as the algorithms behind them are derived from the Davis-Putnam-Longeman-Loveland Procedure [50].

From an engineering perspective, it is elegant to reduce the size of the resolution tree returned by the solver since the reduction can be easily integrated into the configurator as an independent module. Surprisingly, there is little work on such reduction techniques. Greshman et al. propose to reduce the resolution tree by repeated calls to the SAT solver [80]. Amjad proposes a compression technique for resolution proofs [4]. The same is addressed by Sinz but with the assumption that the proof is obtained from a SAT solver [185]. Bar-Ilan et al. propose a linear algorithm for reduction of a resolution tree [12].

Another concept related to this discussion are the *Minimal Unsatisfiable Cores (MUS)*, defined as the subsets of clauses of an unsatisfiable CNF formula that are unsatisfiable but any of its supersets are satisfiable. Clearly, a proof that an MUS is unsatisfiable, is also a proof of unsatisfiability of the whole formula. If the MUS is small, then the proof is likely to be small as well.

A popular technique for approximating an MUS was proposed by Zhang and Malik [205], which calls the solver again on the proof that it has just produced. As the proof is an unsatisfiable formula, the solver will find it unsatisfiable and produce a new proof (possibly smaller than the original one). This process is iterated until a call to the solver does not affect the size of the proof. The algorithm QuickXplain proposed by Junker is another popular algorithm to obtain an MUS or an approximation thereof [111]. Both algorithms, Zhang and Malik's and QuickXplain, are practically interesting because they can be added

on top of a solver, i.e., the solver serves as a black-box.

There is a large body of research on finding MUSs and the interested reader is referred to an overview of these techniques by Gregoire et al. [84].

10.2 Related Work for Logic Optimizations

Formula preprocessing techniques are well known in the SAT solver community (e.g., Several authors focus on binary clauses, i.e., implications, in the context of satisfiability. Aspvall et al. show that the satisfiability of a CNF formula that contains only binary clauses can be decided in polynomial time [7]. Both del Val and Brafman show that satisfiability testing can be sped up if BISs, i.e., strongly connected components in the implication graph (see Definition 10), are collapsed [54, 32]. Bacchus incorporates support for binary clauses into Davis Putnam procedure by generalizing the notion of resolution [10]

Freuder introduces the concept of *interchangeable values* in the context of CSP [76]. Values are interchangeable if changing one to the other in a solution always yields a solution. This concept is dual to BIS: interchangeable values let us treat a set of values as one value, BISs let us treat a set of variables as one variable.

Several authors focus on techniques that preprocess a formula before it is given to a SAT solver with the goal of making the solver more efficient on the preprocessed formula. Eén and Biere design a technique based on resolution and it is implemented in the tool SatElite [60]. Fourdrinoy et al. design a technique that eliminates redundant clauses in polynomial time [71]. For an overview of simplification techniques for SAT see [133, 134].

To the best knowledge of the author, there has been no research concerned with reconstructing the proofs from the proof obtained from the optimized formula as is done in Section 5.2.

10.3 Completing a Configuration Process

The work on automated completion of a configuration process is quite limited. Some authors propose to apply a solver in order to complete the configuration in *some* way, i.e., the scenario **A** as in Section 6.1 [21].

Krebs et al. study the problem of how to help the user to finish the configuration process with the means of machine learning [120]. The learning algorithm tries to identify a certain *plan* in the decisions of the user made so far. Once identified, this plan is used to automatically suggest to them steps in the configuration process.

Section 6.4.2 shows that a configuration process may be facilitated if there is some notion of preference on the possible configurations. Preference as such has been studied by many researchers and a full discussion on this topic is beyond the scope of this dissertation. For an overview of formalization of preference in logic see [55], in CSP see [172], and in human interaction see [166].

Amilhastre et al. enable the user to assign preferences to the decisions. These preferences then affect the form of explanations [3]. Freuder et al. propose a *preference-based configurator* in which different constraints may have different preferences and the configurator recomputes these preferences after a user decision has been made [78]. The preferences are numbers between 0 and 1 and are combined using the minimum operator. The proposed configurator does not find an optimum as it considers *soft arc consistency* (only pairs of variables are taken into account).

Junker studies the application of preference in the context of non-interactive configuration of a CSP [110]. The problem he tackles is that the user is interested in some solutions of the CSP and the approach Junker suggests is that the user may specify preferences of how these solutions are searched for using a dedicated language.

10.3.1 Computing Dispensable Variables

As shown in Chapter 6, dispensable variables are closely related to propositional circumscription. Originally, circumscription was devised by McCarthy as a form of reasoning in the '80s [143]. Minker showed the relation between propositional circumscription and GCWA in '82 (a relation shown in Section 6.2.4) [152].

From computational perspective, an important complexity result is that propositional circumscription is Π_2^P-complete [63]. Moreover, even the query corresponding to determine whether a variable is dispensable is Π_2^P-complete [63, Lemma 3.1].

Recall that the algorithm MinModels used in Section 6.3 to compute dispensable variables invokes the SAT solver as many times as there are minimal models (see Claim 10). Lonc and Truszczynski estimate that the number of minimal models of a CNF formula is exponential [130].

Giunchiglia and Maratea modify DPLL procedure in an analogous way to obtain an algorithm similar to the one presented in Section 6.3.1 in order to solve problems with preference [81]. The problem of finding one minimal model is a specific instance of the problem solved by these authors. Indeed, Giunchiglia and Maratea refer to the problem of finding one minimal model as Min-One$_{\subseteq}$. Extending this work, Di Rosa et al. focus on enumerating the most preferred solutions [171]. The approach is analogous to the one used in Section 6.3, i.e.,

finding one minimal model and adding a clause that blocks all models than the one just found. In fact, this approach has been used as early as 1996 by Castell et al. in similar context [37].

The main difference between the work on minimal model enumeration referenced in the above paragraph (Castell et al. and Di Rosa et al.) and Section 6.3 is that these authors modify the DPLL (in the form of a SAT solver) to return multiple solutions while the approach of Section 6.3 is to invoke the procedure multiple times. The advantage of modifying the procedure is that some information is reused for finding further solutions (such as learned clauses). The advantage of invoking the procedure is that the algorithm can be used with different instances of the procedure. Namely, the algorithm can be used with different SAT solvers or with any procedure that finds a single minimal model of a formula.

Avin and Zohary solve the problem of propositional circumscription by enumerating minimal models [8]. This enumeration is done by exploring possible models of the formula and testing whether they are indeed minimal; an ordering of the models is exploited in order to avoid testing all possible models.

Kavvadias et al. show a correspondence between the enumeration of all minimal models and traversals of a hyper-graph. Further they devise an output-polynomial algorithm for enumerating all minimal models of a formula in a conjunctive normal form that contains only binary clauses [115].

Lee and Lin [124] and show that a propositional circumscription can be translated into *disjunctive logic programs* with the stable model semantics (see [127]). Oikarinen and Janhunen evolve this idea and provide an implementation connected to an existing implementations of disjunctive logic programming [100, 160].

For an overview of calculi and approaches to computation for circumscription see at chapter of the Handbook of Automated Reasoning dedicated to this topic [58]. In a broader perspective, both GCWA and circumscription belong to the domain of *non-monotonic reasoning*, see an overview of research in this area compiled by Minker [153] for more details.

10.4 Building a Configurator

The communication within the reasoning backend starts at the user interface and ends at the SAT solver. Whenever the user requests an explanation, the information flows in the opposite direction as well. A similar patterns of data flow appear in *round-trip engineering*. In round-trip engineering two forms of the same information are edited and are kept synchronized. Whenever one of the forms changes, the other form needs to be automatically updated. This is

common in model driven engineering where models and source code need to be synchronized when the system evolves. For an overview, see a survey by Czarnecki et al. [45]. In some approaches, the transition from models to source code is done in multiple steps [145, 119] similarly to the way the semantic translation is carried out in the reasoning backend. Foster et al. use a technique called *lenses* to keep different representations of the same information synchronized [70].

In the semantic translation, however, the translation needs to take into account the semantic characteristic of the translated elements, whereas the round-tripping approaches are typically syntactic.

In terms of architecture and workings the reasoning backend is similar to the verification condition generators. Such generators take a program annotated with specifications and produce a formula that is valid if the program meets the specifications; this formula is called the *verification condition*. This formula is send to a prover, typically a first-order logic or a higher-order logic prover [67, 13, 40, 68, 73].

When the verification condition is proven not to hold, the generator displays to the user locations in the program where the specification is violated. A technique commonly used for providing this kind of feedback is based on labeling parts of the verification condition. The prover provides a counter-example that contains the relevant labels and the generator tracks of these labels back to the source code [126].

10.5 SAT Solvers and their Applications

This section provides pointers to other areas of research where SAT solving plays an important role. Since it is out of scope of this dissertation to provide a detailed overview of all these areas, only several exemplars were chosen. An overview of SAT applications can be found in a recent article by Silva [141] and an overview of SAT techniques is found in a study by Gu et al. [87].

The potential of SAT solvers started to be apparent in the late 90's, demonstrated by such solvers as GRASP [138, 139], SATO [204], or Chaff [158]. Since then the list of solvers and their power has been growing as can be seen from the history of the SAT competition [176]. A numerous techniques contributed to the improvements of efficiency in SAT solvers and the interested reader is referred to the Handbook of satisfiability for further details [182].

One of the significant domains for application of SAT solvers is the verification and testing of electrical circuits. Biere et al. used a SAT solver to model check properties of a microprocessor [23]. Similarly, Bjesse et al. search for bugs in an Alpha microprocessor [25]. The application of SAT solvers to model checking was further studied by Abdulla et al. [1]. Silva and Sakallah show the

application of SAT solvers in electronic design automation [183]. Sheeran et al. verify safety properties using induction [181]. More recently, Dolby et al. applied SAT solvers in the context of software verification [59].

Another important type of application of SAT solvers is as underlying engines in provers for non-propositional logics. Barrett et al. show that certain first-order formulas can be proven by a SAT solver by incremental translation [14]. Satisfiability Modulo Theories (SMT) solvers, such as Z3 or fx7, hinge on a built-in SAT solver [53, 157]. Lin and Zhao show that answer set programming can be realized using a SAT solver [128]. Wang et al. show that description logic can be decided using a SAT solver [198].

Recently, SAT solvers were used for problems that are not naturally expressed in logic; such as cryptography [154], context-free grammars [9], or evolutionary biology [28].

Chapter 11

Conclusions and Future Work

The main motivation of this dissertation is to build a case for using satisfiability (SAT) solvers to implement interactive configurators. Within a configurator, a SAT solver is utilized in a *lazy* fashion, i.e., all computation is carried out *during* the configuration process. This contrasts with the precompilation techniques that do most of the computation *before* the configuration process starts. While the precompilation techniques have the advantage of guaranteed response time, we argue that the lazy approach is more practical.

To support our argument, we show that the lazy approach *scales uniformly* without the need of heuristics specific to a particular instance and it is possible to accurately estimate the response time (Chapter 8). Our approach is also practical from the implementation point of view. The core algorithms needed for implementing a configurator based on our principles are lightweight. The reason why that is the case stems from the fact that most of the computation takes place in the SAT solver. Hence, by using a SAT solver as the underlying technology, we are taking advantage of the outstanding progress that took place in the research on satisfiability in the past two decades. This research not only resulted in significant improvements in the runtimes of solvers but also in a number of solvers that are freely available and easy to integrate as third-party components.

Another benefit that comes from the use of SAT solvers is their ability to provide resolution-based proofs of unsatisfiability. We demonstrate that such proofs enable us to produce *more informative explanations* than those that are produced by precompilation approaches. In particular, the explanations contain parts of the instance being configured and dependencies between them

as opposed to just user decisions (Chapter 4).

While the concept of interactive configuration is well-known, the study of *completion* of a configuration process is neglected in the existing literature. The dissertation addresses this gap by identifying different scenarios of completion of a process and formally establishes the notion of "making a choice between number of options". Further, we show how a SAT solver is used to identify variables that can be eliminated without making a choice for the user (Chapter 6).

In conclusion, the dissertation validates the thesis that SAT solvers are better for implementing Boolean interactive configurators than precompilation-based approaches by demonstrating scalability and overall practicality of the SAT-based approach.

In the following sections we point out the potential for future research. Sections 11.1, 11.3, and 11.4 discuss opportunities for further research from the technical perspective while Section 11.5 makes more general suggestions for the research in interactive configuration as a whole.

11.1 Types of Constraints

This dissertation is concerned with interactive configuration of instances that are captured in propositional logic. A natural direction of future work is to extend this work with support for *non-propositional constraints*. An exemplar of non-propositional constraints are *pseudo-Boolean constraints*. Pseudo-Boolean constraints are linear inequalities where variables have the values 0 or 1 (corresponding to False and True, respectively). The following example illustrates the usefulness of such constraints.

Example 40. *Let us have three components. Whether a component is selected or not is represented by a variable c_i, for $i \in 1..3$. Any Boolean constraint can be expressed as a linear equation. For instance, to express that the first component requires the second component we write $c_1 < c_2$. Let us have a constant price for each component p_1, p_2, and p_3, respectively. The requirement that the total price must be at most K is expressed by the equation $p_1c_1 + p_2c_2 + p_3c_2 \leq K$.*

Any set of pseudo-Boolean constraints can be translated into a Boolean formula. In principle, such translation may blow up in size. However, there are numerous successful encoding techniques that enable performing this translation efficiently [11, 184, 62, 137].

These encoding techniques give us the means to support pseudo-Boolean constraints in interactive configuration while relying on the techniques described in this dissertation: applying any of the encodings gives us a Boolean formula to which all the described algorithms apply. This suggests the following research

question: *What other types of constraints can be configured using a SAT solver?*

11.2 Syntactic Optimizations

The BIS-optimization (Chapter 5) did show some promising results but not of large significance (Chapter 8). This is an encouragement to study other optimizations techniques that are known the SAT community (see Section 10.2). While the effect of such optimizations in SAT solving has been evaluated, the effect in interactive configuration is expected to be different because the cost of the optimization is amortized over multiple calls to a SAT solver.

Another topic important in the context of syntactic optimizations is the proof reconstruction. In the BIS-optimization we provided a way how to reconstruct a proof for a formula from a proof of its optimized counterpart. We have observed that the size of the reconstructed proof may vary greatly depending on how the optimization is carried out. This observation leads to the following research question: *How to provide proofs in the presence of syntactic optimizations, and, what methods can be employed to reduce the size of the reconstructed proof?*

An interesting phenomenon occurred in some of the instances' explanation sizes. When the BIS-optimization was used, in some cases the reconstructed explanation was smaller than when the optimization was not used (Section 8.5). This indicates that the algorithms used for computing explanations are sensitive to the size of the input formula and sometimes perform significantly better when the formula is smaller. This yields the following research question: *What effect do syntactic optimizations have on proof sizes?*

11.3 Dispensable Variables

In Chapter 6, dispensable variables are defined as those variables that can be eliminated without making a choice for the user (Definition 16). The empirical evaluation shows that the technique for computing dispensable variables shown in Section 6.3 works well on smaller instances, but scales poorly on the larger ones (Section 8.7). This encourages further investigation of methods for computing dispensable variables. Due to the relation of dispensable variables to propositional circumscription (Section 6.2.4), there are a number of options to consider [58]. Hence, we propose the following research question: *How to efficiently compute the set of dispensable variables?*

11.4 Implementation

The separation of concerns provided by the architecture of the configurator proved to be useful during the development. In particular, it enabled experimenting with the implementation by adding or replacing one of the translators forming the chain of translation. However, at this moment it is the programmer who assembles the configurator by modifying the source code. One way of making this more elegant would be to provide a configuration interface for the configurator and use some model-driven approach to assemble the configurator. There is a number of technologies enabling such infrastructure. For instance, the AHEAD Tool Suite [17] or aspect-oriented programming [125].

11.5 Hindsight and Future

In hindsight, the dissertation is more theory-driven than application-driven. Namely, the empirical evaluation was carried out at the end of the work. The disadvantage of this approach is demonstrated by the BIS-optimization. The optimization proved to be useful in several instances, however, in these instances the response time is already good from the user perspective without the optimization. Consequently, the dissertation demonstrates the usefulness of the technique only weakly. Rather, the optimization serves as a proof-of-concept that shows how to integrate a syntactic optimization into a configurator. If some preliminary empirical evaluation were performed at the beginning of the work, the choice of the syntactic optimization would be different.

In a similar regard, little attention was paid to what users of configurators really need in practice. In fact, there appears to be a large gap between the research on variability modeling and on application thereof in practice, i.e., a gap between state-of-the-art and state-of-the-practice.

One such issue comes from models of large sizes. If the configured model contains thousands of variables or more, the model becomes hard to comprehend. We conjecture that it is possible to use automated reasoning to help users in such cases. For instance, the most important variables could be automatically identified according to some criteria.

Another important topic is explanations. The dissertation describes how to build explanations from resolution trees, and clearly, these explanations are more informative than explanations comprising only user decisions. However, we do not understand to what extent such explanations help users in practice. Moreover, there may exist multiple resolution trees that are semantically equivalent. If that is the case, which of these trees are more intuitive for the user than the others? More generally speaking, the research questions are: *How are*

explanations useful in practice? and *How to present mechanical proofs to a human?*

In conclusion, the research on interactive configuration of variability models needs to get closer to what is needed in practice. The facets most painfully missing in this context are *benchmarks* and *user-oriented experiments* evaluating the existing techniques.

Bibliography

[1] Parosh Aziz Abdulla, Per Bjesse, and Niklas Eén. Symbolic reachability analysis based on SAT-Solvers. In Susanne Graf and Michael I. Schwartzbach, editors, *TACAS*, volume 1785 of *Lecture Notes in Computer Science*, pages 411–425. Springer, 2000.

[2] NIST/SEMATECH e-handbook of statistical method. http://www.itl.nist.gov/div898/handbook/.

[3] Jérôme Amilhastre, Hélène Fargier, and Pierre Marquis. Consistency restoration and explanations in dynamic CSPs application to configuration. *Artificial Intelligence*, 135(1-2):199–234, 2002.

[4] Hasan Amjad. Compressing propositional refutations. *Electronic Notes in Theoretical Computer Science*, 185:3–15, 2007.

[5] Henrik Reif Andersen, Tarik Hadzic, and David Pisinger. Interactive cost configuration over decision diagrams. *Journal of Artificial Intelligence Research*, 2010.

[6] *First Workshop on Analyses of Software Product Lines (ASPL '08)*, 2008. Available at http://www.isa.us.es/aspl08.

[7] Bengt Aspvall, Michael F. Plass, and Robert Endre Tarjan. A linear-time algorithm for testing the truth of certain quantified boolean formulas. *Inf. Process. Lett.*, 8(3):121–123, 1979.

[8] Chen Avin and Rachel Ben-Eliyahu-Zohary. Algorithms for computing X-minimal models. In *Logic Programming and Nonmonotonic Reasoning (LPNMR)*. Springer, 2001.

[9] Roland Axelsson, Keijo Heljanko, and Martin Lange. Analyzing context-free grammars using an incremental SAT solver. In *ICALP '08: Proceedings of the 35th international colloquium on Automata, Languages and Programming, Part II*, pages 410–422, Berlin, Heidelberg, 2008. Springer-Verlag.

[10] Fahiem Bacchus. Enhancing davis putnam with extended binary clause reasoning. In *AAAI/IAAI*, pages 613–619, 2002.

[11] Olivier Bailleux and Yacine Boufkhad. Efficient CNF encoding of Boolean cardinality constraints. In Francesca Rossi, editor, *CP*, volume 2833 of *Lecture Notes in Computer Science*, pages 108–122. Springer, 2003.

[12] Omer Bar-Ilan, Oded Fuhrmann, Shlomo Hoory, Ohad Shacham, and Ofer Strichman. Linear-time reductions of resolution proofs. In Hana Chockler and Alan J. Hu, editors, *Haifa Verification Conference*, volume 5394 of *Lecture Notes in Computer Science*, pages 114–128. Springer, 2008.

[13] Michael Barnett, Bor-Yuh Evan Chang, Robert DeLine, Bart Jacobs, and K. Rustan M. Leino. Boogie: A modular reusable verifier for object-oriented programs. In Frank S. de Boer, Marcello M. Bonsangue, Susanne Graf, and Willem P. de Roever, editors, *FMCO*, volume 4111 of *Lecture Notes in Computer Science*, pages 364–387. Springer, 2005.

[14] Clark W. Barrett, David L. Dill, and Aaron Stump. Checking satisfiability of first-order formulas by incremental translation to SAT. In Ed Brinksma and Kim Guldstrand Larsen, editors, *CAV*, volume 2404 of *Lecture Notes in Computer Science*, pages 236–249. Springer, 2002.

[15] Don Batory. Feature models, grammars, and propositional formulas. In H. Obbink and K. Pohl, editors, *Proceedings of the 9th International Software Product Line Conference (SPLC '05)*, volume 3714 of *Lecture Notes in Computer Science*. Springer-Verlag, September 2005.

[16] Don Batory and Sean O'Malley. The design and implementation of hierarchical software systems with reusable components. *ACM Transactions on Software Engineering and Methodology*, 1992.

[17] Don Batory, Jacob Neal Sarvela, and Axel Rauschmayer. Scaling step-wise refinement. *IEEE Transactions on Software Engineering*, 30(6), 2004.

[18] Don S. Batory, Vivek Singhal, Jeff Thomas, Sankar Dasari, Bart J. Geraci, and Marty Sirkin. The GenVoca model of software-system generators. *IEEE Software*, 11(5):89–94, 1994.

[19] Eli Ben-Sasson and Avi Wigderson. Short proofs are narrow—resolution made simple. *Journal of the ACM*, 48(2):149–169, 2001.

[20] David Benavides, Sergio Segura, and Antonio Ruiz-Cortés. Automated analysis of feature models 20 years later: A literature review. *Inf. Syst.*, 35(6):615–636, 2010.

[21] David Benavides, Pablo Trinidad, and Antonio Ruiz-Cortés. Automated reasoning on feature models. In Pastor Oscar and João Falcão e Cunha, editors, *Proceedings of 17th International Conference on Advanced Information Systems Engineering (CAiSE 05)*, volume 3520 of *Lecture Notes in Computer Science*. Springer-Verlag, 2005.

[22] Thorsten Berger, Steven She, Rafael Lotufo, Andrzej Wasowski, and Krzysztof Czarnecki. Variability modeling in the real: A perspective from the operating systems domain. In *25th IEEE/ACM International Conference on Automated Software Engineering (ASE'10)*, September 2010.

[23] Armin Biere, Edmund M. Clarke, Richard Raimi, and Yunshan Zhu. Verifiying safety properties of a Power PC microprocessor using symbolic model checking without BDDs. In Nicolas Halbwachs and Doron Peled, editors, *CAV*, volume 1633 of *Lecture Notes in Computer Science*, pages 60–71. Springer, 1999.

[24] Armin Biere and Carla P. Gomes, editors. *Theory and Applications of Satisfiability Testing - SAT 2006, 9th International Conference, Seattle, WA, USA, August 12-15, 2006, Proceedings*, volume 4121 of *Lecture Notes in Computer Science*. Springer, 2006.

[25] Per Bjesse, Tim Leonard, and Abdel Mokkedem. Finding bugs in an alpha microprocessor using satisfiability solvers. In *CAV '01: Proceedings of the 13th International Conference on Computer Aided Verification*, pages 454–464, London, UK, 2001. Springer-Verlag.

[26] Beate Bollig and Ingo Wegener. Improving the variable ordering of OBDDs is NP-complete. *IEEE Transactions on Computers*, 45(9):993–1002, 1996.

[27] Béla Bollobás, Christian Borgs, Jennifer T. Chayes, Jeong Han Kim, and David Bruce Wilson. The scaling window of the 2-SAT transition. *Random Structures and Algorithms*, 18(3):201–256, 2001.

[28] Maria Luisa Bonet and Katherine St. John. Efficiently calculating evolutionary tree measures using SAT. In Oliver Kullmann, editor, *SAT*, volume 5584 of *Lecture Notes in Computer Science*, pages 4–17. Springer, 2009.

[29] Goetz Botterweck, Denny Schneeweiss, and Andreas Pleuss. Interactive techniques to support the configuration of complex feature models. In *1st International Workshop on Model Driven Product Line Engineering (MDPLE)*, 2009.

[30] Goetz Botterweck, Steffen Thiel, Daren Nestor, Saad bin Abid, and Ciarán Cawley. Visual tool support for configuring and understanding Software Product Lines. In *The 12th International Software Product Line Conference (SPLC)*, 2008.

[31] Aaron R. Bradley and Zohar Manna. *The Calculus of Computation: Decision Procedures with Applications to Verification*. Springer-Verlag, 2008.

[32] Ronen I. Brafman. A simplifier for propositional formulas with many binary clauses. *IEEE Transactions on Systems, Man, and Cybernetics, Part B*, 34(1):52–59, 2004.

[33] Yuri Breitbart, Harry B. Hunt III, and Daniel J. Rosenkrantz. On the size of binary decision diagrams representing Boolean functions. *Theor. Comput. Sci.*, 145(1&2):45–69, 1995.

[34] Randal E. Bryant. Graph-based algorithms for Boolean function manipulation. *IEEE Transactions on Computers*, 35(8), 1986.

[35] Jerry R. Burch, Edmund M. Clarke, Kenneth L. McMillan, David L. Dill, and L. J. Hwang. Symbolic model checking: 10^{20} states and beyond. In *LICS*, pages 428–439. IEEE Computer Society, 1990.

[36] Marco Cadoli and Maurizio Lenzerini. The complexity of closed world reasoning and circumscription. In *The Eighth National Conference on Artificial Intelligence (AAAI)*, 1990.

[37] Thierry Castell, Claudette Cayrol, Michel Cayrol, and Daniel Le Berre. Using the Davis and Putnam procedure for an efficient computation of preferred models. In *In ECAI '96*, 1996.

[38] Moses Charikar, Chandra Chekuri, To-Yat Cheung, Zuo Dai, Ashish Goel, Sudipto Guha, and Ming Li. Approximation algorithms for directed Steiner problems. In *Proceedings of the 9th annual ACM-SIAM symposium on Discrete algorithms*, pages 192–200. Society for Industrial and Applied Mathematics Philadelphia, PA, USA, 1998.

[39] Paul Clements and Linda Northrop. *Software Product Lines: Practices and Patterns*. Addison–Wesley Publishing Company, 2002.

[40] Ernie Cohen, Markus Dahlweid, Mark A. Hillebrand, Dirk Leinenbach, Michał Moskal, Thomas Santen, Wolfram Schulte, and Stephan Tobies. VCC: A practical system for verifying concurrent C. In Stefan Berghofer, Tobias Nipkow, Christian Urban, and Makarius Wenzel, editors, *TPHOLs*,

volume 5674 of *Lecture Notes in Computer Science*, pages 23–42. Springer, 2009.

[41] Stephen A. Cook. The complexity of theorem-proving procedures. In *STOC '71: Proceedings of the third annual ACM symposium on Theory of computing*, pages 151–158, New York, NY, USA, 1971. ACM.

[42] Stephen A. Cook and Robert A. Reckhow. The relative efficiency of propositional proof systems. *Journal of Symbolic Logic*, 44(1):36–50, 1979.

[43] Krzysztof Czarnecki, Thomas Bednasch, Peter Unger, and Ulrich Eisenecker. Generative programming for embedded software: An industrial experience report. In Don Batory, Charles Consel, and Walid Taha, editors, *Proceedings of the ACM SIGPLAN/SIGSOFT Conference on Generative Programming and Component Engineering (GPCE '02)*, volume 2487 of *Lecture Notes in Computer Science*. Springer Berlin/Heidelberg, October 2002.

[44] Krzysztof Czarnecki and Ulrich W. Eisenecker. *Generative Programming – Methods, Tools, and Applications*. Addison–Wesley Publishing Company, 2000.

[45] Krzysztof Czarnecki, J. Nathan Foster, Zhenjiang Hu, Ralf Lämmel, Andy Schürr, and James F. Terwilliger. Bidirectional transformations: A cross-discipline perspective. In Richard F. Paige, editor, *ICMT*, volume 5563 of *Lecture Notes in Computer Science*, pages 260–283. Springer, 2009.

[46] Krzysztof Czarnecki, Simon Helsen, and Ulrich Eisenecker. Formalizing cardinality-based feature models and their staged configuration. Technical Report 04-11, Department of Electrical and Computer Engineering, University of Waterloo, Canada, April 2004.

[47] Krzysztof Czarnecki, Simon Helsen, and Ulrich Eisenecker. Staged configuration using feature models. In R. L. Nord, editor, *Proceedings of Third International Conference on Software Product Lines (SPLC '04)*, volume 3154 of *Lecture Notes in Computer Science*. Springer-Verlag, August 2004.

[48] Krzysztof Czarnecki and Andrzej Wasowski. Feature diagrams and logics: There and back again. In Kellenberger [116].

[49] Adnan Darwiche and Pierre Marquis. A knowledge compilation map. *Journal of Artificial Intelligence Research*, 2002.

[50] Martin Davis, George Logemann, and Donald Loveland. A machine program for theorem-proving. *Communication of the ACM*, 5(7):394–397, 1962.

[51] Merijn de Jonge and Joost Visser. Grammars as feature diagrams. Presented at the Generative Programming Workshop 2002, Austin, Texas, April 2002.

[52] Johan de Kleer. An Assumption-Based TMS. *Artificial Intelligence*, 28(2):127–162, 1986.

[53] Leonardo Mendonça de Moura and Nikolaj Bjørner. Z3: An efficient SMT solver. In C. R. Ramakrishnan and Jakob Rehof, editors, *TACAS*, volume 4963 of *Lecture Notes in Computer Science*, pages 337–340. Springer, 2008.

[54] Alvaro del Val. Simplifying binary propositional theories into connected components twice as fast. *Lecture Notes in Computer Science*, pages 392–406, 2001.

[55] James Delgrande, Torsten Schaub, Hans Tompits, and Kewen Wang. A classification and survey of preference handling approaches in nonmonotonic reasoning. *Computational Intelligence*, 20(2):308–334, 2004.

[56] Frank DeRemer and Hans Kron. Programming-in-the large versus programming-in-the-small. In *Proceedings of the international conference on Reliable software*, page 121. ACM, 1975.

[57] Edsger W. Dijkstra. *A Discipline of Programming*. Prentice–Hall, Inc., 1976.

[58] Jürgen Dix, Ulrich Furbach, and Ilkka Niemelä. Nonmonotonic reasoning: Towards efficient calculi and implementations. In Andrei Voronkov and Alan Robinson, editors, *Handbook of Automated Reasoning*, chapter 19, pages 1241–1354. Elsevier Science, 2001.

[59] Julian Dolby, Mandana Vaziri, and Frank Tip. Finding bugs efficiently with a SAT solver. In *ESEC-FSE '07: Proceedings of the the 6th joint meeting of the European software engineering conference and the ACM SIGSOFT symposium on The foundations of software engineering*, pages 195–204, New York, NY, USA, 2007. ACM.

[60] Niklas Eén and Armin Biere. Effective preprocessing in SAT through variable and clause elimination. In *Theory and Applications of Satisfiability Testing (SAT '05)*. Springer, 2005.

[61] Niklas Eén and Niklas Sörensen. An extensible SAT-solver. In *Theory and Applications of Satisfiability Testing (SAT '03)*. Springer-Verlag, 2003.

[62] Niklas Eén and Niklas Sörensson. Translating Pseudo-Boolean constraints into SAT. *JSAT*, 2(1-4):1–26, 2006.

[63] Thomas Eiter and Georg Gottlob. Propositional circumscription and extended closed world reasoning are Π_2^P-complete. *Theoretical Computer Science*, 1993.

[64] Mathieu Estratat and Laurent Henocque. Parsing languages with a configurator. In Ramon López de Mántaras and Lorenza Saitta, editors, *ECAI*, pages 591–595. IOS Press, 2004.

[65] Hélène Fargier and Marie-Catherine Vilarem. Compiling CSPs into tree-driven automata for interactive solving. *Constraints*, 9(4):263–287, 2004.

[66] Feature model repository web page. http://fm.gsdlab.org.

[67] Jean-Christophe Filliâtre and Claude Marché. The Why/Krakatoa/Caduceus platform for deductive program verification. In Werner Damm and Holger Hermanns, editors, *CAV*, volume 4590 of *Lecture Notes in Computer Science*, pages 173–177. Springer, 2007.
http://why.lri.fr/.

[68] Cormac Flanagan, K. Rustan M. Leino, Mark Lillibridge, Greg Nelson, James B. Saxe, and Raymie Stata. Extended static checking for Java. In *PLDI*, pages 234–245, 2002.

[69] Kenneth D. Forbus and Johan de Kleer. *Building Problem Solvers*. MIT Press, Cambridge, MA, USA, 1993.

[70] J. Nathan Foster, Michael B. Greenwald, Jonathan T. Moore, Benjamin C. Pierce, and Alan Schmitt. Combinators for bi-directional tree transformations: a linguistic approach to the view update problem. In *POPL '05: Proceedings of the 32nd ACM SIGPLAN-SIGACT symposium on Principles of programming languages*, pages 233–246, New York, NY, USA, 2005. ACM.

[71] Olivier Fourdrinoy, Éric Grégoire, Bertrand Mazure, and Lakhdar Sais. Eliminating redundant clauses in sat instances. In *CPAIOR*, pages 71–83. Springer, 2007.

[72] William B. Frakes and Kyo Kang. Software reuse research: Status and future. *IEEE Trans. Software Eng.*, 31(7):529–536, 2005.

[73] Frama-C homepage.
http://frama-c.cea.fr/.

[74] Eugene C. Freuder. Synthesizing constraint expressions. *Communication of the ACM*, 21(11):958–966, 1978.

[75] Eugene C. Freuder. A sufficient condition for backtrack-free search. *Journal of the ACM*, 29(1):24–32, 1982.

[76] Eugene C. Freuder. Eliminating interchangeable values in constraint satisfaction problems. In *AAAI*, pages 227–233, 1991.

[77] Eugene C. Freuder, Tom Carchrae, and J. Christopher Beck. Satisfaction guaranteed. In *Proceedings of the IJCAI-2003 Workshop on Configuration*, 2003.

[78] Eugene C. Freuder, Chavalit Likitvivatanavong, Manuela Moretti, Francesca Rossi, and Richard J. Wallace. Computing explanations and implications in preference-based configurators. In Barry O'Sullivan, editor, *International Workshop on Constraint Solving and Constraint Logic Programming*, pages 76–92. Springer, 2002.

[79] Eugene C. Freuder, Chavalit Likitvivatanavong, and Richard J. Wallace. Explanation and implication for configuration problems. In *Proceedings of the 17th Int. Joint Conf. on Artificial Intelligence (IJCAI '01), Workshop on Configuration*, 2001.

[80] Roman Gershman, Maya Koifman, and Ofer Strichman. Deriving small unsatisfiable cores with dominators. In Thomas Ball and Robert B. Jones, editors, *CAV*, volume 4144 of *Lecture Notes in Computer Science*, pages 109–122. Springer, 2006.

[81] Enrico Giunchiglia and Marco Maratea. Solving optimization problems with DLL. In *Proceeding of the 2006 conference on ECAI '06*. IOS Press, 2006.

[82] gnuplot—a portable command-line driven graphing utility http://www.gnuplot.info/.

[83] C.P. Gomes, B. Selman, and H. Kautz. Boosting combinatorial search through randomization. In *Proceedings of the National Conference on Artificial Intelligence*, 1998.

[84] Éric Grégoire, Bertrand Mazure, and Cédric Piette. On approaches to explaining infeasibility of sets of boolean clauses. In *The 20th IEEE International Conference on Tools with Artificial Intelligence (ICTAI '08)*, November 2008.

[85] Peter Gregory, Maria Fox, and Derek Long. A new empirical study of weak backdoors. In *International Conference on Principles and Practice of Constraint Programming*, pages 618–623, 2008.

[86] Jan Frisco Groote and Hans Zantema. Resolution and binary decision diagrams cannot simulate each other polynomially. *Discrete Applied Mathematics*, 130(2):157–171, 2003. The Renesse Issue on Satisfiability.

[87] Jun Gu, Paul W. Purdom, John Franco, and Benjamin W. Wah. Algorithms for the satisfiability (SAT) problem: A survey. *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, 35, 1997.

[88] Tarik Hadzic. *Constraint Processing Over Decision Diagrams*. PhD thesis, IT University of Copenhagen, 2007.

[89] Tarik Hadzic and Henrik R. Andersen. A BDD-based polytime algorithm for cost-bounded interactive configuration. In *Proceedings of the National Conference on Artificial Intelligence*. Menlo Park, CA; Cambridge, MA; London; AAAI Press; MIT Press; 1999, 2006.

[90] Tarik Hadzic and Henrik Reif Andersen. Interactive configuration based on linear programming. Technical report, IT University of Copenhagen, 2005.

[91] Tarik Hadzic and Henrik Reif Andersen. Interactive reconfiguration in power supply restoration. In van Beek [195], pages 767–771.

[92] Tarik Hadzic, Sathiamoorthy Subbarayan, Rune M. Jensen, Henrik R. Andersen, Jesper Møller, and Henrik Hulgaard. Fast backtrack-free product configuration using a precompiled solution space representation. In *Proceedings of the International Conference on Economic, Technical and Organizational aspects of Product Configuration Systems, DTU*, pages 131–138, 2004.

[93] Esben Rune Hansen and Henrik R. Andersen. Interactive configuration with regular string constraints. In *AAAI '07: Proceedings of the 22nd national conference on Artificial intelligence*, pages 217–223. AAAI Press, 2007.

[94] Esben Rune Hansen and Peter Tiedemann. Improving the performance of interactive configuration with regular string constraints. In *Proceedings of the 2008 20th IEEE International Conference on Tools with Artificial Intelligence-Volume 01*, pages 3–10. IEEE Computer Society, 2008.

[95] Emmanuel Hebrard, Brahim Hnich, Barry O'Sullivan, and Toby Walsh. Finding diverse and similar solutions in constraint programming. In Manuela M. Veloso and Subbarao Kambhampati, editors, *AAAI*, pages 372–377. AAAI Press / The MIT Press, 2005.

[96] C.A.R. Hoare. An axiomatic basis for computer programming. *Communications of the ACM*, 12:576–580, 1969.

[97] Peter Höfner, Ridha Khedri, and Bernhard Möller. Feature algebra. In *Proceedings of the 14th International Symposium on Formal Methods (FM)*, volume 4085 of *Lecture Notes in Computer Science*. Springer-Verlag, 2006.

[98] Charles Hoover. A methodology for determining response time baselines. In *Int. CMG Conference*, pages 85–94. Computer Measurement Group, 2006.

[99] Katsuro Inoue, Reishi Yokomori, Hikaru Fujiwara, Tetsuo Yamamoto, Makoto Matsushita, and Shinji Kusumoto. Component rank: relative significance rank for software component search. In *ICSE '03: Proceedings of the 25th International Conference on Software Engineering*, pages 14–24, Washington, DC, USA, 2003. IEEE Computer Society.

[100] Tomi Janhunen and Emilia Oikarinen. Capturing parallel circumscription with disjunctive logic programs. In José Júlio Alferes and João Alexandre Leite, editors, *JELIA*, volume 3229 of *Lecture Notes in Computer Science*, pages 134–146. Springer, 2004.

[101] Mikoláš Janota and Joseph Kiniry. Reasoning about feature models in higher-order logic. In Kellenberger [116].

[102] Mikoláš Janota. PidiSAT. Available at http://kind.ucd.ie/products/opensource/Config/releases/.

[103] Mikoláš Janota. Do SAT solvers make good configurators? In ASPL08 [6]. Available at http://www.isa.us.es/aspl08.

[104] Mikoláš Janota and Goetz Botterweck. Formal approach to integrating feature and architecture models. In *Proceeding of Fundamental Approaches to Software Engineering (FASE)*, Lecture Notes in Computer Science. Springer-Verlag, 2008.

[105] Mikoláš Janota, Goetz Botterweck, Radu Grigore, and Joao Marques-Silva. How to complete an interactive configuration process? In *Proceeding of 36th International Conference on Current Trends in Theory and Practice of Computer Science (SOFSEM)*. Springer-Verlag, 2010.

[106] Mikoláš Janota, Victoria Kuzina, and Andrzej Wasowski. Model construction with external constraints: An interactive journey from semantics to syntax. In Krzysztof Czarnecki, Ileana Ober, Jean-Michel Bruel, Axel

Uhl, and Markus Völter, editors, *MoDELS*, volume 5301 of *Lecture Notes in Computer Science*, pages 431–445. Springer, 2008.

[107] JatSAT http://www-verimag.imag.fr/~cotton/jat/.

[108] JavaBDD. http://javabdd.sourceforge.net/.

[109] Rune M. Jensen. CLab: A C++ library for fast backtrack-free interactive product configuration. In *Principles and Practice of Constraint Programming (CP '04)*. Springer, 2004.

[110] Ulrich Junker. Preference programming for configuration. In *Workshop on Configuration*, 2001.

[111] Ulrich Junker. QuickXplain: Conflict detection for arbitrary constraint propagation algorithms. In *Workshop on Modelling and Solving problems with constraints IJCAI '01*, 2001.

[112] Toni Jussila, Carsten Sinz, and Armin Biere. Extended resolution proofs for symbolic SAT solving with quantification. In Biere and Gomes [24], pages 54–60.

[113] Andreas Kaiser and Wolfgang Küchlin. Detecting inadmissible and necessary variables in large propositional formulae. In *Proceedings of International Joint Conference on Automated Reasoning: IJCAR 2001 (Short Papers)*, 2001.

[114] Kyo C. Kang, Sholom G. Cohen, James A. Hess, William E. Novak, and A. Spencer Peterson. Feature-oriented domain analysis (FODA), feasibility study. Technical Report CMU/SEI-90-TR-021, SEI, Carnegie Mellon University, November 1990.

[115] Dimitris J. Kavvadias, Martha Sideri, and Elias C. Stavropoulos. Generating all maximal models of a Boolean expression. *Information Processing Letters*, 74(3-4):157–162, 2000.

[116] Patrick Kellenberger, editor. *Software Product Lines*. IEEE Computer Society, 2007.

[117] Gregor Kiczales, Erik Hilsdale, Jim Hugunin, Mik Kersten, Jeffrey Palm, and William G. Griswold. An overview of AspectJ. In *Proceedings of the 15th European Conference on Object-Oriented Programming (ECOOP)*, volume 2072 of *Lecture Notes in Computer Science*, page 110. Springer-Verlag, 2001.

[118] Philip Kilby, John K. Slaney, Sylvie Thiébaux, and Toby Walsh. Backbones and backdoors in satisfiability. In *AAAI Conference on Artificial Intelligence*, pages 1368–1373, 2005.

[119] Joseph R. Kiniry and Fintan Fairmichael. Ensuring consistency between designs, documentation, formal specifications, and implementations. In *The 12th International Symposium on Component Based Software Engineering (CBSE '09)*, 2009.

[120] Thorsten Krebs, Thomas Wagner, and Wolfgang Runte. Recognizing user intentions in incremental configuration processes. In *Workshop on Configuration*, 2003.

[121] Charles W. Krueger. Software reuse. *Computing Surveys*, 24(2):131–183, 1992.

[122] Vipin Kumar. Algorithms for constraint-satisfaction problems: A survey. *AI Magazine*, 13(1):32–44, 1992.

[123] Gary T. Leavens, Albert L. Baker, and Clyde Ruby. *Behavioral Specifications of Business and Systems*, chapter JML: A Notation for Detailed Design, pages 175–188. Kluwer Academic Publishing, 1999.

[124] Joohyung Lee and Fangzhen Lin. Loop formulas for circumscription. In Deborah L. McGuinness and George Ferguson, editors, *AAAI*, pages 281–286, 2004.

[125] Kwanwoo Lee, Goetz Botterweck, and Steffen Thiel. Feature-modeling and aspect-oriented programming: Integration and automation. In Haeng-Kon Kim and Roger Y. Lee, editors, *SNPD*, pages 186–191. IEEE Computer Society, 2009.

[126] K. Rustan M. Leino, Todd Millstein, and James B. Saxe. Generating error traces from verification-condition counterexamples. *Science of Computer Programming*, 2004.

[127] Vladimir Lifschitz. Foundations of logic programming. *Principles of Knowledge Representation*, pages 69–127, 1996.

[128] Fangzhen Lin and Yuting Zhao. ASSAT: computing answer sets of a logic program by SAT solvers. *Artif. Intell.*, 157(1-2):115–137, 2004.

[129] Jia Liu, Don Batory, and Srinivas Nedunuri. Modeling interactions in feature oriented software designs. In *Proceedings of ICFI*, 2005.

[130] Zbigniew Lonc and Mirosław Truszczyński. Computing minimal models, stable models, and answer sets. In Catuscia Palamidessi, editor, *ICLP*, volume 2916 of *Lecture Notes in Computer Science*, pages 209–223. Springer-Verlag, 2003.

[131] Claudio Lottaz, Ruth Stalker, and Ian Smith. Constraint solving and preference activation for interactive design. *Artificial Intelligence for Engineering Design, Analysis and Manufacturing (AI EDAM)*, 12(01):13–27, 1998.

[132] Donald W. Loveland. *Automated Theorem Proving: A Logical Basis*. North-Holland Publishing Co., 1978.

[133] Inês Lynce and João P. Marques Silva. The puzzling role of simplification in propositional satisfiability. In *Proceedings of the EPIA Workshop on Constraint Satisfaction and Operational Research Techniques for Problem Solving*, pages 73–86, 2001.

[134] Inês Lynce and João P. Marques Silva. Probing-based preprocessing techniques for propositional satisfiability. In *ICTAI*, pages 105–. IEEE Computer Society, 2003.

[135] Alan K. Mackworth. Consistency in networks of relations. *Artificial intelligence*, 8(1):99–118, 1977.

[136] Jeppe Nejsum Madsen. Methods for interactive constraint satisfaction. Master's thesis, University of Copenhagen, 2003.

[137] João P. Marques and Silva Inês Lynce. Towards robust CNF encodings of cardinality constraints. In *Principles and practice of constraint programming–CP '07*. Springer, 2007.

[138] João P. Marques-Silva and Karem A. Sakallah. GRASP—a new search algorithm for satisfiability. In *ICCAD '96: Proceedings of the 1996 IEEE/ACM international conference on Computer-aided design*, pages 220–227, Washington, DC, USA, 1996. IEEE Computer Society.

[139] João P. Marques-Silva and Karem A. Sakallah. GRASP: A search algorithm for propositional satisfiability. *IEEE Transactions on Computers*, 48(5):506–521, 1999.

[140] João P. Marques-Silva. MUSER. Available upon request from the author jpms@ucd.ie.

[141] João P. Marques-Silva. Practical applications of Boolean satisfiability. In *Proceedings of the 9th International Workshop on Discrete Event Systems. WODES 08*, 2008.

[142] João P. Marques-Silva. Minimal unsatisfiability: Models, algorithms and applications (invited paper). In *ISMVL*, pages 9–14. IEEE Computer Society, 2010.

[143] John McCarthy. Circumscription—a form of non-monotonic reasoning. *Artificial Intelligence*, 13:27–39, 1980.

[144] Kenneth L. McMillan. *Symbolic Model Checking*. Kluwer Academic Publishing, 1993.

[145] Nenad Medvidovic, Alexander Egyed, and David S. Rosenblum. Round-trip software engineering using UML: From architecture to design and back. In *Proceedings of the Second International Workshop on Object-Oriented Reengineering (WOOR 99), Toulouse, France*, 1999.

[146] Nenad Medvidovic and Richard N. Taylor. A classification and comparison framework for software architecture description languages. *IEEE transactions on Software Engineering*, 2000.

[147] Marcílio Mendonça, Andrzej Wasowski, and Krzysztof Czarnecki. SAT-based analysis of feature models is easy. In Dirk Muthig and John D. McGregor, editors, *SPLC*, volume 446 of *ACM International Conference Proceeding Series*, pages 231–240. ACM, 2009.

[148] Marcílio Mendonça, Andrzej Wasowski, Krzysztof Czarnecki, and Donald D. Cowan. Efficient compilation techniques for large scale feature models. In Yannis Smaragdakis and Jeremy G. Siek, editors, *GPCE*, pages 13–22. ACM, 2008.

[149] Bertrand Meyer. Applying "design by contract". *Computer*, 25(10):40–51, 1992.

[150] Edward Minieka. The centers and medians of a graph. *Operations Research*, 25(4):641–650, 1977.

[151] MiniSAT. http://minisat.se/.

[152] Jack Minker. On indefinite databases and the Closed World Assumption. In *Proceedings of the 6th Conference on Automated Deduction*. Springer-Verlag, 1982.

[153] Jack Minker. An overview of nonmonotonic reasoning and logic programming. *J. Log. Program.*, 17(2/3&4):95–126, 1993.

[154] Ilya Mironov and Lintao Zhang. Applications of SAT solvers to cryptanalysis of hash functions. In Biere and Gomes [24], pages 102–115.

[155] David G. Mitchell, Bart Selman, and Hector J. Levesque. Hard and easy distributions of SAT problems. In *AAAI*, pages 459–465, 1992.

[156] Rémi Monasson, Riccardo Zecchina, Scott Kirkpatrick, Bart Selman, and Lidror Troyansk. Determining computational complexity from characteristic 'phase transitions'. *Nature*, 400:133–137, July 1999.

[157] Michał Moskal. *Satisfiability Modulo Software*. PhD thesis, Institute of Computer Science, University of Wrocław, 2009.

[158] Matthew W. Moskewicz, Conor F. Madigan, Ying Zhao, Lintao Zhang, and Sharad Malik. Chaff: Engineering an efficient SAT solver. In *The 39th Design Automation Conference (DAC '01)*, 2001.

[159] Barry O'Callaghan, Barry O'Sullivan, and Eugene C. Freuder. Generating corrective explanations for interactive constraint satisfaction. In Peter van Beek, editor, *CP*, volume 3709 of *Lecture Notes in Computer Science*, pages 445–459. Springer, 2005.

[160] Emilia Oikarinen and Tomi Janhunen. circ2dlp - translating circumscription into disjunctive logic programming. In Chitta Baral, Gianluigi Greco, Nicola Leone, and Giorgio Terracina, editors, *LPNMR*, volume 3662 of *Lecture Notes in Computer Science*, pages 405–409. Springer, 2005.

[161] Barry O'Sullivan. Interactive constraint-aided conceptual design. *AI EDAM*, 16(4):303–328, 2002.

[162] Christos H. Papadimitriou. *Computational Complexity*. John Wiley and Sons Ltd., 1994.

[163] Alexandre Papadopoulos and Barry O'Sullivan. Relaxations for compiled over-constrained problems. In Peter J. Stuckey, editor, *CP*, volume 5202 of *Lecture Notes in Computer Science*, pages 433–447. Springer, 2008.

[164] Bernard Pargamin. Extending cluster tree compilation with non-boolean variables in product configuration: A tractable approach to preference-based configuration. In *Proceedings of the IJCAI-03 Workshop on Configuration, Acapulco, Mexico*, 2003.

[165] David L. Parnas. On the design and development of program families. *IEEE Transactions on Software Engineering*, 1976.

[166] Bart Peintner, Paolo Viappiani, and Neil Yorke-Smith. Preferences in interactive systems: Technical challenges and case studies. *AI Magazine*, 29(4):13, 2009.

[167] PicoSAT http://fmv.jku.at/picosat.

[168] Christian Prehofer. Feature-oriented programming: A fresh look at objects. In *ECOOP*. Springer-Verlag, 1997.

[169] Spyros Reveliotis. *An Introduction to Linear Programming and the Simplex Algorithm*. School of Industrial and Systems Engineering, Georgia Institute of Technology, 1997. Availabe at http://www2.isye.gatech.edu/~spyros/LP/LP.html.

[170] cnf2bdd—available upon request from the author radugrigore@gmail.com.

[171] Emanuele Di Rosa, Enrico Giunchiglia, and Marco Maratea. Solving satisfiability problems with preferences.

[172] Francesca Rossi, K. Brent Venable, and Toby Walsh. Preferences in constraint satisfaction and optimization. *AI Magazine*, 29(4):58–68, 2008.

[173] Stuart Russell and Peter Norvig. *Artificial Intelligence: A Modern Approach*. Prentice Hall, second edition, 2003.

[174] S^2T^2 Configurator. http://download.lero.ie/spl/s2t2/.

[175] Daniel Sabin and Rainer Weigel. Product configuration frameworks—a survey. *IEEE Intelligent Systems*, 13(4):42–49, 1998.

[176] SAT competition. http://www.satcompetition.org/.

[177] SAT Race. http://baldur.iti.uka.de/sat-race-2010/.

[178] SAT4J. http://www.sat4j.org/.

[179] Pierre-Yves Schobbens, Patrick Heymans, and Jean-Christophe Trigaux. Feature diagrams: A survey and a formal semantics. In *Proceeding of 14th IEEE International Requirements Engineering Conference (RE '06)*. IEEE Computer Society, 2006.

[180] Sergio Segura. Automated analysis of feature models using Atomic Sets. In ASPL08 [6]. Available at http://www.isa.us.es/aspl08.

[181] Mary Sheeran, Satnam Singh, and Gunnar Stålmarck. Checking safety properties using induction and a SAT-Solver. In Warren A. Hunt Jr. and Steven D. Johnson, editors, *FMCAD*, volume 1954 of *Lecture Notes in Computer Science*, pages 108–125. Springer, 2000.

[182] João P. Marques Silva, Inês Lynce, and Sharad Malik. Conflict-driven clause learning sat solvers. In Armin Biere, Marijn Heule, Hans van Maaren, and Toby Walsh, editors, *Handbook of Satisfiability*, volume 185 of *Frontiers in Artificial Intelligence and Applications*, pages 131–153. IOS Press, 2009.

[183] João P. Marques Silva and Karem A. Sakallah. Boolean satisfiability in electronic design automation. In *DAC*, pages 675–680, 2000.

[184] Carsten Sinz. Towards an optimal CNF encoding of Boolean cardinality constraints. In van Beek [195], pages 827–831.

[185] Carsten Sinz. Compressing propositional proofs by common subproof extraction. In Roberto Moreno-Díaz, Franz Pichler, and Alexis Quesada-Arencibia, editors, *EUROCAST*, volume 4739 of *Lecture Notes in Computer Science*, pages 547–555. Springer, 2007.

[186] Carsten Sinz and Armin Biere. Extended resolution proofs for conjoining BDDs. *Computer Science–Theory and Applications*, pages 600–611, 2006.

[187] Sathiamoorthy Subbarayan. Integrating CSP decomposition techniques and BDDs for compiling configuration problems. In *Proceedings of the CP-AI-OR*. Springer-Verlag, 2005.

[188] Sathiamoorthy Subbarayan and Henrik R. Andersen. Linear functions for interactive configuration using join matching and CSP tree decomposition. In *Papers from the Configuration Workshop at IJCAI'05*, 2005.

[189] Sathiamoorthy Subbarayan, Rune M. Jensen, Tarik Hadzic, Henrik R. Andersen, Henrik Hulgaard, and Jesper Møller. Comparing two implementations of a complete and backtrack-free interactive configurator. In *Proceedings of the CP-04 Workshop on CSP Techniques with Immediate Application*, pages 97–111, 2004.

[190] Robert E. Tarjan. Depth-first search and linear graph algorithms. *SIAM Journal on Computing*, 1:146, 1972.

[191] Seinosuke Toda. PP is as hard as the polynomial-time hierarchy. *SIAM J. Comput.*, 20(5):865–877, 1991.

[192] G. S. Tseitin. On the complexity of derivation in propositional calculus. *Studies in constructive mathematics and mathematical logic*, 2(115-125):10–13, 1968.

[193] Tomás E. Uribe and Mark E. Stickel. Ordered binary decision diagrams and the Davis-Putnam procedure. In Jean-Pierre Jouannaud, editor, *CCL*, volume 845 of *Lecture Notes in Computer Science*, pages 34–49. Springer, 1994.

[194] Leslie G. Valiant. The complexity of computing the permanent. *Theoretical Computer Science*, 8(2):189 – 201, 1979.

[195] Peter van Beek, editor. *Principles and Practice of Constraint Programming - CP 2005, 11th International Conference, CP 2005, Sitges, Spain, October 1-5, 2005, Proceedings*, volume 3709 of *Lecture Notes in Computer Science*. Springer, 2005.

[196] Erik Roland van der Meer, Andrzej Wasowski, and Henrik Reif Andersen. Efficient interactive configuration of unbounded modular systems. In Hisham Haddad, editor, *SAC*, pages 409–414. ACM, 2006.

[197] Nageshwara Rao Vempaty. Solving Constraint Satisfaction Problems using Finite State Automata. In *AAAI*, pages 453–458, 1992.

[198] C. Wang, F. Ivančić, M. Ganai, and A. Gupta. Deciding separation logic formulae by sat and incremental negative cycle elimination. In *Logic for Programming, Artificial Intelligence, and Reasoning*, pages 322–336. Springer, 2005.

[199] Jules White, Douglas C. Schmidt, David Benavides, Pablo Trinidad, and Antonio Ruiz Cortés. Automated diagnosis of product-line configuration errors in feature models. In *SPLC*, pages 225–234. IEEE Computer Society, 2008.

[200] Niklaus Wirth. Program development by stepwise refinement. *Communication of the ACM*, 14(4):221–227, 1971.

[201] Bwolen Yang, Randal E. Bryant, David R. O'Hallaron, Armin Biere, Olivier Coudert, Geert Janssen, Rajeev K. Ranjan, and Fabio Somenzi. A performance study of BDD-Based model checking. In Ganesh Gopalakrishnan and Phillip J. Windley, editors, *FMCAD*, volume 1522 of *Lecture Notes in Computer Science*, pages 255–289. Springer, 1998.

[202] Haining Yao and Letha Etzkorn. Towards a semantic-based approach for software reusable component classification and retrieval. In *ACM-SE*

42: Proceedings of the 42nd annual Southeast regional conference, pages 110–115, New York, NY, USA, 2004. ACM.

[203] Yunwen Ye and Gerhard Fischer. Information delivery in support of learning reusable software components on demand. In *Proceedings of the 7th international conference on Intelligent user interfaces*, page 166. ACM, 2002.

[204] Hantao Zhang. SATO: An efficient propositional prover. In *CADE-14: Proceedings of the 14th International Conference on Automated Deduction*, pages 272–275, London, UK, 1997. Springer-Verlag.

[205] Lintao Zhang and Sharad Malik. Extracting small unsatisfiable cores from unsatisfiable boolean formulas. In *Proceedings of SAT*, 2003.

[206] Lintao Zhang and Sharad Malik. Validating SAT solvers using an independent resolution-based checker: practical implementations and other applications. In *Design, Automation and Test in Europe Conference and Exhibition (DATE)*, 2003.

[207] Wei Zhang, Haiyan Zhao, and Hong Mei. A propositional logic-based method for verification of feature models. In *Formal Methods and Software Engineering*. Springer, 2004.

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